Structural induction
Counting I

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This lecture finishes structural induction (in section 4.3 in Rosen) and starts the topic of counting, covering sections 5.1 and 5.3.

1 Announcements

Midterm coming up on Wednesday. If you have a conflict, you should have received email from me with the details of the conflict exam. If not, please contact me right away.

2 Structural induction with 2D points

Last class, we saw examples of "structural" induction proofs using trees. Structural induction is used to prove a claim about a set T of objects which is defined recursively. Instead of having an explicit induction variable n, our proof follows the structure of the recursive definition.

- Show the claim holds for the base case(s) of the definition of T
- For the recursive cases of T's definition, show that if the claim holds for the smaller/input objects, then it holds for the larger/output objects.

To see how this works on a set of things that aren't trees, consider the following recursive definition of a set S of 2D points:

1. $(3, 5) \in S$
2. If $(x, y) \in S$, then $(x + 2, y) \in S$
3. If $(x, y) \in S$, then $(-x, y) \in S$
4. If $(x, y) \in S$, then $(y, x) \in S$

What's in S? Starting with the pair specified in the base case $(3, 5)$, we use rule 3 to add $(-3, 5)$. Rule 2 then allows us to add $(-1, 5)$ and then $(1, 5)$. If we apply rule 2 repeatedly, we see that S contains all pairs of the form $(2n + 1, 5)$ where x is a natural number. By rule 4, $(5, 2n + 1)$ must also be in S for every natural number n.

We then apply rules 2 and 3 in the same way, to show that $(2m+1, 2n+1)$ is in S for every natural numbers m and n.

We've now shown, albeit somewhat informally, that every pair with odd coordinates is a member of S. But does every member of S have odd coordinates?

To show that all members of S have odd coordinates, we use structural induction.

> Proof by structural induction that all elements of S have both coordinates odd.
>
> Base: Both coordinates of $(3, 5)$ are odd.
>
> Induction: Suppose that both coordinates of (x, y) are odd. We need to show that both coordinates of $(x + 2, y)$, $(-x, y)$, and (y, x) are odd. But this is (even in the context of this course), obvious.

This proof would be complicated to write with standard induction. It's possible to find a standard induction variable n, but it's done in a somewhat obscure way: the "size" of an element x in S is the number of times you have to apply the recursive rules in S's definition in order to show that x is in S.

3 Introduction to counting

Many applications require counting, or estimating, the size of a finite set. For small examples, you can just list all the elements. For examples with a simple structure, you could probably improvise the right answer. But when it's not so obvious, some techniques are often helpful.

4 Product rule

> **The product rule**: if you have p choices for pinning down one feature of the object and then q choices for a second feature, and your options for the second feature don't depend on what you chose for the first one, then you have pq options total.

So if T-shirts can come in 4 colors and 3 sizes, there are $4 \cdot 3 = 12$ types of T-shirts.

In general, if the objects are determined by a sequence of independent decisions, the number of different objects is the product of the number of options for each decision. So if the T-shirts come in 4 colors, 5 sizes, and 2 types of necklines, there are $4 \cdot 5 \cdot 2 = 40$ types of shirts.

Yup. That's pretty much what you thought. Nothing hard here.

5 The sum rule, inclusion/exclusion

> **The sum rule**: suppose your task can be done in one of two ways, which are mutually exclusive. If the first way has p choices and the second way has q choices, then you have $p+q$ choices for how to do the task.

Example: it's late evening and you want to watch TV. You have 37 programs on cable, 57 DVD's on the shelf, and 12 movies stored in I-tunes. So you have $37 + 57 + 12 = 106$ options for what to watch.

Weeeellll, maybe. This analysis assumes that there is no overlap between the movies available on the three media. If any movies are in more than one collection, this will double-count them. So we would need to subtract off the number that have been double-counted. For example, if the only overlap is that 2 movies are on I-tunes and also on DVD, you would have only $(37 + 57 + 12) - 2 = 104$ options.

The formal name for this correction is the "Inclusion-Exclusion Principle".

Suppose you have two sets A and B. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

We can use this basic formula to derive the formula for three sets A, B, and C:

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|) \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|
\end{aligned}$$

6 Combining these two rules

Let's see what happens on a more complex example. Suppose S contains all 5-digit decimal numbers that start with 2 one's or end in 2 zeros, where we don't allow leading zeros. How many numbers does S contain?

Let T be the set of 5-digit numbers starting in 2 one's. We know the first two digits and we have three independent choices (10 options each) for the last three. So there are 1000 numbers in T.

Let R be the set of 5-digit numbers ending in 2 zeros. We have 9 options for the first digit, since it can't be zero. We have 10 options each for the second and third digits, and the last two are fixed. So we have 900 numbers in R.

What's the size of $T \cap R$? Numbers in this set start with 2 one's and end with 2 zeros, so the only choice is the middle digit. So it contains 10 numbers. So

$$|S| = |T| + |R| - |T \cap R| = 1000 + 900 - 10 = 1890$$

7 Non-independent decisions

Now, suppose that describing our set of possibilities involves a sequence of decisions that aren't independent, i.e. where later decisions depend on earlier ones. Dependencies can create a lot of very hard-to-analyze situations. However, there is a range of standard situations for which nice formulas exist.

For example, suppose that we have 7 Scrabble tiles, all different from one another, and we want to form a 4-letter word. We have 7 choices for the first letter in the word. But then we have only 6 choices for the second letter, because we've used up one time. So the number of 4-letter words we can form is $7 \cdot 6 \cdot 5 \cdot 4 = 840$. In math jargon, each possible word is called a *4-permutation* of the set of 7 tiles.

To get a sense of the general formula, notice what happens if we want to form a word using all 7 letters. We then have $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways to do it. This is just 7!.

In general, suppose we have a set S of n objects. Then a *permutation* of S is a way of putting the elements of S into an order. There are $n!$ ways to do this, because we have n choices for which element to put first, then $n-1$ choices for which to put second, and so on.

Now, suppose that we only want to pick k objects from S, but the order still matters to us. This is called a k-permutation of S. Then there are $n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$ different k-permutations . This number is called $P(n,k)$.

8 Combinations

The permutations formula applies when we care about the order in which we are selecting the objects, e.g. we are putting them into an arrangement or choosing them for a series of different roles. If we simply want to select a subset of the objects, we need a different formula, which we'll see next class.