Son of Functions

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This lecture finishes covering the main concepts involving functions, though we'll see more example proofs on Friday.

1 Announcements

Remember to bring your ID to Wednesday's exam. Also notice that the exams from previous term were held in class, so expect our exam to be somewhat longer (but nowhere near twice as long).

Office hours Th/Fri this week are cancelled, because there's no homework due. Bring any last-minute questions to lectures Wednesday, or come to Wednesday office hours (regular and extra) 11-12 (Viraj and Samer) and 3:30 (approx) to 5 (Chen and Adair).

2 Recap

Last lecture, we saw that functions are defined with a specific declared domain (set of input values) and co-domain (set of legit output values). The image of a function is the set of actual values produced if you run all the input values through the function. So the image is a subset of the co-domain, but the two might not be equal.

A function $f : A \rightarrow B$ is onto (surjective) if $\forall y \in B, \exists x \in A, f(x) = y$. That is, the image is all of the co-domain.

For example, the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 2$ is onto. But the function $g : \mathbb{R} \rightarrow \mathbb{R}$ $g(x) = x^2$ is not onto, because it never produces negative outputs. g would be onto if we had defined its co-domain to be only the non-negative reals.

In this class, most examples of non-onto functions look like cases where you could have defined them to be onto, but the author just didn't feel like setting up the co-domain precisely. Or, when it comes to computer programs, perhaps the compiler declarations don't allow you to be sufficiently precise, e.g. there's no distinct type for non-negative floating point numbers.

In some applications, however, it's critical that certain functions not be onto. For example, in graphics or certain engineering applications, we may wish to map out or draw a curve in 2D space. The whole point of the curve is that it occupies only part of 2D and it is surrounded by whitespace. These curves are often specified "parametrically," using functions that map into, but not onto, 2D.

For example, we can specify a (unit) circle as the image of a function $f : [0, 1] \to \mathbb{R}^2$ defined by $f(x) = (\cos 2\pi x, \sin 2\pi x)$. If you think of the input values as time, then f shows the track of a pen or robot as it goes around the circle. The cosine and sine are the x and y coordinates of its position. The 2π multiplier simply puts the input into the right range so that we'll sweep exactly once around the circle (assuming that sine and cosine take their inputs in radians).

3 Warning about variations in terminology

It used to be that people used the term "range" to refer to the co-domain. However, formal mathematics has standardized on the term "co-domain" for the declared set of possible output values for the function. When the term "range" is used, it is as a synonym for "image," i.e. the actual output values produced when you feed in all possible input values. (Rosen uses "range" with this meaning.)

We'll try to stick carefully to the newer convention in this class. **Please avoid the term range.** But be aware that older authors and authors outside math/CS may use the terms differently.

4 One-to-one

Last lecture, we had just started discussing the concept of one-to-one.

A function is *one-to-one* if it never assigns two input values to the same output value. That is

$$\forall x, y \in A, x \neq y \rightarrow f(x) \neq f(y)$$

or, equivalently,

$$\forall x, y \in A, f(x) = f(y) \rightarrow x = y$$

(These two versions are equivalent because they are the contrapositives of one another.)

In these two versions of the definition, notice that when you choose x and y, they don't have to be different (math jargon: "distinct") values. In normal English, if you give different names to two objects, the listener is expected to understand that they are different. By contrast, mathematicians always mean you to understand that they might be different but they might be the same object.

For example, let $g : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $g(x) = 2x$. g is one-to-one.

A classic example of a function that's not one-to-one is the absolute value function $|x|$. More precisely, it's not one-to-one if the declared domain is the reals or the integers. It is one-to-one if we restrict the domain to the natural numbers.

5 Pre-images and inverses

Suppose that $f : A \rightarrow B$ is a function from A to B. If we pick a value x in A, then the corresponding output value $f(x)$ is called the image of x. If we pick a value $y \in B$, then $x \in A$ is a *pre-image* of y if $f(x) = y$. Notice that I said **a** pre-image of y, not **the** pre-image of y. For a one-to-one function like g, each element of the co-domain has exactly one pre-image. But this isn't true for functions that aren't one-to-one. For example, if f is the absolute value function from the reals to the reals, the output value 2 has two pre-images: 2 and -2.

6 Proving that a function is one-to-one

Claim 1 *Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f(x) = 3x + 7$. f is one-to-one.*

Let's prove this using our definition of one-to-one.

Proof: We need to show that for every integers x and y, $f(x) = f(y) \rightarrow x = y$.

So, let x and y be integers and suppose that $f(x) = f(y)$. We need to show that $x = y$.

We know that $f(x) = f(y)$. So, substituting in our formula for f, $3x + 7 = 3y + 7$. So $3x = 3y$ and therefore $x = y$, by high school algebra. This is what we needed to show.

When we pick x and y at the start of the proof, notice that we haven't specified whether they are the same number or not. Mathematical convention leaves this vague, unlike normal English where the same statement would strongly suggest that they were different.

You may have encountered the abbreviation QED at the end of a proof. This stands for "quod erat demonstrandum" which is simply the Latin for "this is what we needed to show." Some people put a little box or a little triangle of 3 dots at the end of the proof. It's good style to have something at the end of your proof which tells the reader that the proof is complete.

7 Proving that a function is onto

Now, consider this claim:

Claim 2 *Define the function g from the integers to the integers by the formula $g(x) = x - 8$. g is onto.*

Proof: We need to show that for every integer y, there is an integer x such that $g(x) = y$.

So, let y be some arbitrary integer. Choose x to be $(y + 8)$. x is an integer, since it's the sum of two integers. But then $g(x) = (y + 8) - 8 = y$, so we've found the required pre-image for y and our proof is done.

Notice that our function f from the last section wasn't onto. Suppose we tried to build a proof that it was.

Proof: We need to show that for every integer y, there is an integer x such that $f(x) = y$.

So, let y be some arbitrary integer. Choose x to be $\frac{(y-7)}{3}$. ...

If f was a function from the reals to the reals, we'd be ok at this point, because x would be a good pre-image for y. However, f's inputs are declared to be integers. For many values of y, $\frac{(y-7)}{3}$ isn't an integer. So it won't work as an input value for f.

8 Composing two functions

Suppose that $f : A \to B$ and $g : B \to C$ are functions. Then $g \circ f$ is the function from A to C defined by $(g \circ f)(x) = g(f(x))$. Depending on the author, this is either called the composition of f and g or the composition of g and f. The idea is that you take input values from A, run them through f, and then run the result of that through g to get the final output value.

Take-home message: when using function composition, look at the author's shorthand notation rather than their mathematical English, to be clear on which function gets applied first.

In this definition, notice that g came first in $(g \circ f)(x)$ and g also comes first in $g(f(x))$. I.e. unlike $f(g(x))$ where f comes first. The trick for remembering this definition is to remember that f and g are in the same order on the two sides of the defining equation.

For example, if we use our functions f and g defined above, the domains and co-domains of both functions are the integers. So we can compose the two functions in both orders.

$$(f \circ g)(x) = f(g(x)) = 3g(x) + 7 = 3(x - 8) + 7 = 3x - 24 + 7 = 3x - 17$$
$$(g \circ f)(x) = g(f(x)) = f(x) - 8 = (3x + 7) - 8 = 3x - 1$$

Notice that the order matters to the output value!

Frequently, the declared domains and co-domains of the two functions aren't all the same, so often you can only compose in one order. For example, consider the function $h : \{strings\} \to \mathbb{Z}$ which maps a string x onto its length in characters. (E.g. $h(\text{Margaret}) = 8$.) Then $f \circ h$ exists but $(h \circ f)$ doesn't exist because f produces numbers and the inputs to h are supposed to be strings.

9 A proof involving composition

Consider this claim:

Claim 3 *For any sets A, B, and C and for any functions $f : A \to B$ and $g : B \to C$, if f and g are injective, then $g \circ f$ is also injective.*

We can prove this with a direct proof, by being systematic about using our definitions and standard proof outlines. First, let's pick some representative objects of the right types and assume everything in our hypothesis.

> Proof: Let A, B, and C be sets. Let $f : A \to B$ and $g : B \to C$ be functions. Suppose that f and g are injective.
>
> We need to show that $g \circ f$ is injective.

To show that $g \circ f$ is injective, we need to pick two elements x and y in its domain, assume that their output values are equal, and then show that x and y must themselves be equal. Let's splice this into our draft proof. Remember that the domain of $g \circ f$ is A and its co-domain is C.

> Proof: Let A, B, and C be sets. Let $f : A \to B$ and $g : B \to C$ be functions. Suppose that f and g are injective.
>
> We need to show that $g \circ f$ is injective. So, choose x and y in A and suppose that $(g \circ f)(x) = (g \circ f)(y)$
>
> We need to show that $x = y$.

Now, we need to apply the definition of function composition and the fact that f and g are each injective:

> Proof: Let A, B, and C be sets. Let $f : A \to B$ and $g : B \to C$ be functions. Suppose that f and g are injective.
>
> We need to show that $g \circ f$ is injective. So, choose x and y in A and suppose that $(g \circ f)(x) = (g \circ f)(y)$
>
> Using the definition of function composition, we can rewrite this as $g(f(x)) = g(f(y))$. Combining this with the fact that g is injective, we find that $f(x) = f(y)$. But, since f is injective, this implies that $x = y$, which is what we needed to show.