Functions

Margaret M. Fleck

19 February 2010

This lecture starts the material in section 2.3 of Rosen. It discusses functions and introduces the concepts of one-to-one and onto.

1 Announcements

Another reminder of the upcoming midterm. Remember to bring your ID. (But we do have a backup plan if one or two of you forget.)

This lecture is brought to you by the number 65535. (This is $2^{16} - 1$, i.e. the largest number you can store in a 16-bit unsigned integer variable.)

Another useful fact is that 1000 is approximately equal to 2^{10}. This is helpful when determining how large a number you will get when trying to access locations in computer memory, especially for big memory sizes.

2 Functions

We all know roughly what functions are, from high school and (if you've taken it) calculus. You've mostly seen functions whose inputs and outputs are numbers, defined by an algebraic formula such as $f(x) = 2x + 3$. We're going to generalize and formalize this idea, so we can talk about functions with other sorts of input and output values.

Suppose that A and B are sets, then a function f from A to B (shorthand: $f : A \rightarrow B$) is an assignment of exactly one element of B (i.e. the output value) to each element of A (i.e. the input value). A is called the *domain* of f and B is called the *co-domain*.

For example, let's define $g : \mathbb{Z} \rightarrow \mathbb{Z}$ by the formula $g(x) = 2x$. The domain and co-domain of this function are both $\mathbb{Z}$.

Notice that the domain and co-domain are part of the definition of the function, just like the input/output type declarations for a function in a programming language. Suppose we define $h : \mathbb{N} \to \mathbb{N}$ such that $h(x) = 2x$. This is a **different function from** g because the declared domain and co-domain are different.

Two functions are equal if they have the same domain, the same co-domain, and assign the same output value to each input value.

The inputs and outputs to functions don't have to be numbers and a function doesn't have to be defined by an algebraic formula. It's sufficient to describe a clear, explicit procedure for finding the output value, given the input value. For example, we can define

$$s : \{\text{CS 173 course staff}\} \to \{\text{letters of the alphabet}\}$$

where $s(x)$ is the first letter in x's name. For example $s(\text{Margaret}) = M$.

For small finite sets like these, we can also just list all the input/output pairs:

Margaret $\mapsto$ M

Viraj $\mapsto$ V

Lucas $\mapsto$ L

Lance $\mapsto$ L

Dan $\mapsto$ D

Samer $\mapsto$ S

Chen $\mapsto$ C

Andrew $\mapsto$ A

Efe $\mapsto$ E

Adair $\mapsto$ A

Rick $\mapsto$ R

We also show this with a drawing like this. [draw something similar to Figures 1 in Rosen 2.3]

Notice that we use $\mapsto$ when we show the output value for a a single input value, but $\rightarrow$ to show the input and output sets for the whole function.

A function is also known as a *map* or *mapping* and we can say that some input value x *maps to* the corresponding output value y.

3 What isn't a function?

Much of the definition of a function is an association of output values with input values. Suppose I give you an association p. When is p not a function?

One possibility is that p doesn't provide an output value for every input value. For example, suppose we defined $p : \mathbb{R} \rightarrow \mathbb{R}$ such that $p(x)$ is the multiplicative inverse of x. That is, $p(x)$ is the integer y such that $xy = 1$. This isn't a function because one input value (zero) doesn't have a corresponding output value.[1]

Notice that this depends critically on what we've declared the domain to be. If we revised the type declaration for p to read $p : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$, then p would be a function.

An association is also not a function if it assigns two different output values to the same input value. Suppose if I want to define the locations of top-5 CS departments. I.e. the domain is $D = \{\text{MIT}, \text{CMU}, \text{UIUC}, \text{Stanford}, \text{Berkeley}\}$ and the do-domain is the set of all cities C. I might define c as:

$$\text{MIT} \mapsto \text{Cambridge}$$

$$\text{CMU} \mapsto \text{Pittsburgh}$$

$$\text{UIUC} \mapsto \text{Urbana}$$

$$\text{UIUC} \mapsto \text{Champaign}$$

$$\text{Berkeley} \mapsto \text{Berkeley}$$

$$\text{Stanford} \mapsto \text{Palo Alto}$$

[1] Almost functions that don't provide an output for every input are sometimes useful and are known as *partial functions*. But we won't use them in this class.

This isn't a function because UIUC is mapped to two distinct output values. When we need to have a function return multiple values, we need to return them as sets. For example, we might define $c : D \to \mathbb{P}(C)$ by:

MIT $\mapsto$ {Cambridge}

CMU $\mapsto$ {Pittsburgh}

UIUC $\mapsto$ {Urbana, Champaign}

Berkeley $\mapsto$ {Berkeley}

Stanford $\mapsto$ {Palo Alto}

Since we've declared that our output values are sets, we have to make them all sets. So we have to map (say) MIT to {Cambridge} rather than to the bare string Cambridge.

Another example of this problem would be the function $h : \mathbb{R}^+ \to \mathbb{R}$ such that $h(x) = \sqrt{x}$ which maps each positive real onto its square root. We could fix this function by stipulating that we mean the positive square root.

4 Images and Onto

Suppose we have a function $f : A \to B$. If x is an element of A, then the value $f(x)$ is also known as the *image* of x. The image of f is the set

$$Im(f) = \{f(x) : x \in A\}$$

f is *onto* if its image is its whole co-domain. Or, equivalently,

$$\forall y \in B, \exists x \in A, f(x) = y$$

For example, our function from course staff to first letters of names isn't onto, because its image is the set $\{A, M, D, L, S, E, C, R, V\}$ which is nowhere near all the letters of the alphabet.

Suppose we define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x + 2$. This function is onto. If we pick any integer y, let x be $y - 2$. Then $f(x) = f(y-2) = (y-2)+2 = y$.

Now suppose we define $g : \mathbb{N} \to \mathbb{N}$ using the same formula $g(x) = x + 2$. g isn't onto, because the values 0 and 1 have no pre-images.

Warning: whether a function is onto depends on how we've declared its domain and co-domain. When we're discussing these properties, it is absolutely critical to declare the input/output types for all the functions you are using.

5 Negating expressions with multiple quantifiers

Let's use our definition of onto as an excuse to think about negating formal statements containing more than one quantifier. Our definition was:

$$\forall y \in B, \exists x \in A, f(x) = y$$

So a function f is not onto if

$$\neg\forall y \in B, \exists x \in A, f(x) = y$$

To negate this, we proceed step-by-step, moving the negation inwards. You've seen all the identities involved, so this is largely a matter of being careful.

$$\neg\forall y \in B, \exists x \in A, f(x) = y$$

$$\equiv \exists y \in B, \neg\exists x \in A, f(x) = y$$

$$\equiv \exists y \in B, \forall x \in A, \neg(f(x) = y)$$

$$\equiv \exists y \in B, \forall x \in A, f(x) \neq y$$

So, if we want to show that f is not onto, we need to find some value y in B, such that no matter which element x you pick from A, $f(x)$ isn't equal to y.

6 Pre-images and one-to-one

A function is *one-to-one* if it never assigns two input values to the same output value. That is

$$\forall x, y \in A, x \neq y \rightarrow f(x) \neq f(y)$$

or, equivalently,

$$\forall x, y \in A, f(x) = f(y) \rightarrow x = y$$

(These two versions are equivalent because they are the contrapositives of one another.)

For example, let $g : \mathbb{Z} \to \mathbb{Z}$ be defined by $g(x) = 2x$. g is one-to-one. But remember our function s that mapped CS 173 instructors to the first letter of their names. It's not one-to-one because Lance and Lucas share the same output value L.

Like onto, one-to-one depends on what you've picked for the domain and co-domain For example, let $g : \mathbb{Z} \to \mathbb{Z}$ be defined by $g(x) = x^2$. g is not one-to-one, because 2 and -2 map onto the same output value. But suppose we define $g' : \mathbb{N} \to \mathbb{Z}$ with the same defining equation $g'(x) = x^2$. g' is now one-to-one, because we've eliminated the possibility of negative inputs.

7 Warning about variations in terminology

The terms "injective" and "surjective" are fancy synonyms for one-to-one and onto. No more and no less. It's important to get used to both versions of each term, because individual mathematicians often alternate, even within a single lecture.

8 Bijections

If a function f is both one-to-one and onto, then each output value has exactly one corresponding input value. So we can invert f, to get an inverse function f^{-1}. A function that is both one-to-one and onto is called a *bijection* or a *one-to-one correspondence*. If f maps from A to B, then f^{-1} maps from B to A.

Notice that one-to-one and onto are like the two properties required to be a function: each input gets at most one value, each input gets at least one value). We're just checking that these conditions hold in the backwards direction.