Sets I

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This lecture presents sets and basic operations on sets (Rosen sections 2.1 and 2.2).

1 Announcements

The quiz solutions are on the web. I expect the quizzes themselves to be handed back in discussion sections next week, though the grades may be available online somewhat sooner.

2 Sets

I'm sure you've all seen the basic ideas of sets and, indeed, we've been using some of them all term. It's time to discuss sets systematically, showing you a useful range of constructions, notation, and special cases. A few operations (e.g. power sets and Cartesian products) are probably unfamiliar to many of you. And we'll see how to do proofs of claims involving sets.

Definition: A set is an unordered collection of objects.

For example, the natural numbers are a set. So are the integers between 3 and 7 (inclusive). So are all the planets in this solar system or all the programs written by students in CS 225 in the last three years. The objects in a set can be anything you want.

The items in the set are called its elements or members. We've already seen the notation for this: $x \in A$ means that x is a member of the set A.

There's three basic ways to define a set:

- describe its contents in mathematical English, e.g. "the integers between 3 and 7, inclusive."
- list all its members, e.g. $\{3, 4, 5, 6, 7\}$
- use so-called set builder notation, e.g. $\{x \in \mathbb{Z} \mid 3 \leq x \leq 7\}$

Set builder notation has two parts separated with a vertical bar (or, by some writers, a colon). The first part names a variable (in this case x) that ranges over all objects in the set. The second part one or more constraints that these objects must satisfy, e.g. $3 \leq x \leq 7$. The type of the variable (integer in our example) can be specified either before or after the vertical bar. The separator (| or :) is often read "such that."

Here's an example of a set containing an infinite number of objects

- "multiples of 7"
- $\{\ldots - 14, -7, 0, 7, 14, 21, 18, \ldots\}$
- $\{x \in \mathbb{Z} \mid x \text{ is a multiple of } 7\}$

We couldn't list all the elements, so we had to use "...". This is only a good idea if the pattern will be clear to your reader. If you aren't sure, use one of the other methods.

If we wanted to use shorthand for "multiple of", it might be confusing to have | used for two different purposes. So it would probably be best to use the colon variant of set builder notation:

$\{x \in \mathbb{Z} : 7 \mid x\}$

3 Things to be careful about

A set is an unordered collection. So $\{1, 2, 3\}$ and $\{2, 3, 1\}$ are two descriptions of the same set.

Each element occurs only once in a set. Or, alternatively, it doesn't matter how many times you write it. So $\{1, 2, 3\}$ and $\{1, 2, 3, 2\}$ also describe the same set.

So sets are very different from ordered pairs such as $(1, 2, 2, 3)$. For ordered pairs, the order of values matters and duplicate elements don't magically collapse. $(1, 2, 2, 3) \neq (1, 2, 3)$ and $(1, 2, 2, 3) \neq (2, 2, 1, 3)$. Therefore, carefully distinguish curly brackets (set) from parentheses (ordered pair).

Put the curly brackets around the list of members when you are defining a set.

The empty set (also called the null set) is a special set that contains no elements. It is written $\emptyset$. Don't write it as $\{\}$, even though that might seem sensible. The notation $\emptyset$ is firmly entrenched in mathematics and not using it will cause readers to think less of your mathematical skill. (It's like making a spelling mistake on your resume.)

The empty set may seem like a pain in the neck. However, computer science applications are full of empty lists, strings of zero length, and the like. It's the kind of special case that all of you (even the non-theoreticians) will spend your life having to watch out for.

A set can contain objects of more than one type, e.g. $\{a, b, 3, 7\}$. A set can also contain sets, e.g. $\{\mathbb{Z}, \mathbb{Q}\}$ is a set containing two infinite sets. $\{\{a, b\}, \{c\}\}$ is a set containing two finite sets.

Because of this, $\{\emptyset\}$ is not the empty set but, rather, a set with one member, which is the empty set.

4 Cardinality, inclusion

If A is a finite set (a set containing only a finite number of objects), then $|A|$ is the number of objects in A. This is also called the **cardinality** of A.

For example, $|\{a, b, 3\}| = 3$. The notation of cardinality also extends to sets with infinitely many members ("infinite sets") such as the integers, but we won't get into the details of that right now.

Notice that the notation $|A|$ might mean set cardinality or it might be the more familiar absolute value. To tell which, figure out what type of object A is. If it's a set, the author meant cardinality. If it's a number, the author meant absolute value.

If the set contains objects with complex structure, the cardinality is the number of top-level objects. Don't flatten out the structure and count the basic objects in them. For example

$$|\{1, \{1\}, \{\{1\}, 3\}\}| = 3$$

The top-level objects are

- 1
- $\{1\}$
- $\{\{1\}, 3\}$

If A and B are sets, then A is a subset of B ($A \subseteq B$) if every element of A is also in B. Or, if you want it formally, $\forall x, x \in A \rightarrow x \in B$. For example, $\mathbb{Q} \subseteq \mathbb{R}$, because every member of the rationals is also a member of the reals.

The notion of subset allows the two sets to be equal. So $A \subseteq A$ is true for any set A. So $\subseteq$ is like $\leq$. If you want to force the two sets to be different (i.e. like $<$), you must say that A is a **proper** subset of B, written $A \subset B$. These two symbols can be reversed, e.g. $B \supseteq A$ means the same as $A \subseteq B$.

The empty set is a subset of any set A. This is an example of vacuous truth. Subset requires that for every object x, if x is an element of the empty set, then x is an element of A. But this if/then statement is considered true because its hypothesis is always false.

5 Set operations

Given two sets A and B, the intersection of A and B ($A \cap B$) is the set containing all objects that are in both A and B. In set builder notation:

$$A \cap B = \{S \mid S \in A \text{ and } S \in B\}$$

Let's set up some sample sets:

- $M = \{\text{egg}, \text{bread}, \text{milk}\}$
- $P = \{\text{milk}, \text{egg}, \text{flour}\}$

Then $M \cap P$ is $\{\text{milk}, \text{egg}\}$.

If the intersection of two sets A and B is the empty set, i.e. the two sets have no elements in common, then A and B are said to be **disjoint**.

The union of sets A and B ($A \cup B$) is the set containing all objects that are in one (or both) of A and B. So $M \cup P$ is $\{\text{milk}, \text{egg}, \text{bread}, \text{flour}\}$.

The set difference of A and B ($A - B$) contains all the objects that are in A but not in B. In this case,

$$M - P = \{\text{bread}\}$$

The complement of a set A ($\overline{A}$) is all the objects that aren't in A. For this to make sense, you need to define your "universal set" (often written U). U contains all the objects of the sort(s) you are discussing. For example, in some discussions, U might be all real numbers. U doesn't contain everything you might imagine putting in a set, because constructing a set that inclusive leads to paradoxes. U is more limited than that. Whenever U is used, you and your reader need to come to an understanding about what's in it.

So, if our universe is all integers, and A contains all the multiples of 3, then $\overline{A}$ is all the integers whose remainder mod 3 is either 1 or 2. $\overline{\mathbb{Q}}$ would be the irrational numbers if our universe is all real numbers. If we had been working with complex numbers, it might be the set of all irrational real numbers plus all the numbers with an imaginary component.

6 Power sets and cartesian products

If A is a set, the powerset of A ($\mathbb{P}(A)$ or 2^A) is the set containing all subsets of A.

For example, suppose that $A = \{1, 2, 3\}$. Then

$$\mathbb{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

If S is finite and contains n elements, then $\mathbb{P}(S)$ contains 2^n elements. Or, in shorthand, $|\mathbb{P}(S)| = 2^{|S|}$. This is the reason behind the alternate notation 2^S for the powerset of S.

Notice that the powerset of A always contains the empty set, regardless of what's in A. $\mathbb{P}(\emptyset) = \{\emptyset\}$.

If A and B are two sets, their Cartesian product ($A \times B$) contains all ordered pairs (x, y) where x is in A and y is in B. That is

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

For example, if $A = \{a, b\}$ and $B = \{1, 2\}$, then

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

Notice that order matters for Cartesian product.

$$B \times A = \{(1, a), (2, a), (1, b), (2, b)\}$$

These sets are typically not equal.

If $|A| = n$ and $|B| = m$, then $|A \times B| = nm$.