Method of Unrolling

- Consider the following inductively defined function:

  $T(1) = 1$

  $\forall n \geq 2,\ T(n) = T(n-1) + 3n$

- Claim: $T(n)$ is $O(n^2)$

- Proof: Suppose n is very large. Then

$$
\begin{aligned}
T(n) &= T(n-1) + 3n \quad \text{(by definition)} \\
&= T(n-2) + 3((n-1) + n) \\
&= T(n-3) + 3((n-2) + (n-1) + n) \\
&\ldots \\
&= T(1) + 3(2 + 3 + \ldots + (n-2) + (n-1) + n) \\
&= 1 + 3\left(\sum_{i=2}^{n} i\right) = 1 + 3\left(\sum_{i=1}^{n} i - 1\right) = 1 + 3\frac{n(n+1)}{2} - 1 = \frac{3}{2}n(n+1)
\end{aligned}
$$

which is $O(n^2)$

Method of Unrolling: Another example

- Consider the following inductively defined function:

  $M(0) = p$

  $\forall n \geq 2,\ M(n) = (1.05)M(n-1)$

- Claim: $T(n)$ is $O(1.05^n)$

- Proof: Suppose n is very large. Then

$$
\begin{aligned}
M(n) &= (1.05)M(n-1) \quad \text{(by definition)} \\
&= (1.05)^2 M(n-2) \\
&= (1.05)^3 M(n-3) \\
&\ldots \\
&= (1.05)^n M(0) \\
&= (1.05)^n p
\end{aligned}
$$

which is $O(1.05^n)$

Sample proofs with Big O

- We have just shown that $M(n)$ is $O(1.05^n)$

- Claim 1: $M(n)$ is $O(2^n)$

- Proof 1: Since $M(n)$ is $O(1.05^n)$, $\exists m \in \mathbf{N}, \exists c > 1, \forall n \geq m, M(n) \leq c(1.05^n)$
  Now $\forall n \in \mathbf{N}$, $1.05^n \leq 2^n$, so: $\exists m \in \mathbf{N}, \exists c > 1, \forall n \geq m, M(n) \leq c(2^n)$
  Hence $M(n)$ is $O(2^n)$

- Claim 2: $M(n)$ is not $\Omega(2^n)$

- Proof 2: Suppose $M(n)$ is $\Omega(2^n)$. Thus 2^n is $O(M(n))$ and so by definition
  $\exists m \in \mathbf{N}, \exists c > 0, \forall n \geq m, \quad 2^n \leq c.M(n) = c.(1.05^n).p$
  So $\exists m \in \mathbf{N}, \exists c > 0, \forall n \geq m, \ (2/1.05)^n \leq c.p$
  So $\exists m \in \mathbf{N}, \exists c > 0, \forall n \geq m, \ n \leq \log_{(2/1.05)}(cp)$
  This is false e.g., when $n \geq \max(m, \log_{(2/1.05)}(cp)+1)$. This proves Claim 2