Another proof by strong induction

- **Claim**: $\forall n \in \mathbf{N}$, Player 2 can win an (n, n) Nim game

- **Proof** by induction on n:

  Base case ($n = 1$): Player 1 can only pick one stick, and Player 2 wins by picking the other stick

  Inductive step: Let $k \in \mathbf{N}$ such that $\forall m \leq k$, Player 2 can win a (k, k) Nim game [strong IH]

  We need to show that Player 2 can win a $(k+1, k+1)$ Nim game. Suppose Player 1 picks j sticks. Assume WLOG that this is from the first pile, so we are left with a $(k+1-j, k+1)$ game. There are two cases:

  Case 1: $j = k+1$, in which case Player 2 wins the $(0, k+1)$ game by picking $j = k+1$ sticks from the second pile

  Case 2: $1 \leq j < k+1$, in which case Player 2 picks j sticks and we are left with a $(k+1-j, k+1-j)$ game, which Player 2 can win (by the strong IH). Hence the statement holds for $k+1$, which completes the proof by (strong) induction.

Inductive Definitions

- An inductive definition (also called a recursive definition) has two parts:
  - a base case (or base cases)
  - an inductive (or recursive) defintion

- *Example* 1: The factorial function $! : \mathbf{N} \to \mathbf{N}$ is defined inductively as:

  $0! = 1$

  $\forall n \geq 1,\ n! = n \times (n-1)!$

- *Example* 2: The n^{th} Fibonacci number is defined inductively as:

  $F_0 = 0$

  $F_1 = 1$

  $\forall n \geq 2,\ F_n = F_{n-1} + F_{n-2}$

- Thus, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, ... etc.

A Proof with Fibonacci Numbers

- **Claim**: $\forall n \in \mathbf{N}$, F_{3n} is even

- Proof by induction on n:

Base case (n = 0): $F_{3\times 0} = F_0 = 0$, which is even

Inductive step: Let $k \in \mathbf{N}$ such that F_{3k} is even [IH]. We need to show that $F_{3(k+1)}$ is even.

$$
\begin{aligned}
\text{Now } F_{3(k+1)} = F_{3k+3} &= F_{3k+2} + F_{3k+1} \\
&= F_{3k+1} + F_{3k} + F_{3k+1} \\
&= F_{3k} + 2F_{3k+1}
\end{aligned}
$$

By the IH F_{3k} is even, and $2F_{3k+1}$ is also even. Hence $F_{3(k+1)}$ is even, which completes the proof by induction.

A False Proof with Fibonacci Numbers

- **"Claim"**: $\forall n \in \mathbf{N},\ F_n \le 2n$

- "Proof" by induction on n:

  Base case ($n = 0$): $F_0 = 0 \le 2 \times 0$

  Inductive step: Let $k \in \mathbf{N}$ such that ($\forall m \le k,\ F_m \le 2m$) [strong IH]. We need to show that $F_{k+1} \le 2(k+1)$.

  Now F_{k+1} $= F_k + F_{k-1}$
  $\le 2k + 2(k-1)$
  $= 4k - 2$

  Error: This applies only when $k+1 \ge 2$, which may not be true! Add a base case ($n = 1$). We can then assume $k \ge 1$, i.e. $k+1 \ge 2$

  ... oops, looks like we're stuck!

Fixing the False Proof

- **Claim**: $\forall n \in \mathbf{N},\ F_n < 2^n$

- Proof by induction on n:

Base case ($n = 0$): $F_0 = 0 < 2^0$
Base case ($n = 1$): $F_1 = 1 < 2^1$

Inductive step: Let $k \geq 1$ such that $(\forall m \leq k,\ F_m < 2^m)$ [strong IH]. We need to show that $F_{k+1} < 2^{k+1}$

$$
\begin{aligned}
\text{Now } F_{k+1} &= F_k + F_{k-1} \quad (\text{since } k+1 \geq 2) \\
&< 2^k + 2^{k-1} \\
&< 2 \times 2^k = 2^{k+1}
\end{aligned}
$$

Hence the statement holds for k+1, which completes the proof by induction.