Example

- **Claim**: $\forall n \in \mathrm{N}, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- **Proof** by induction on *n*:

Base case ($n = 0$): We need to show that $P(0)$ is true i.e., $\sum_{i=1}^{0} i = \frac{0(0+1)}{2}$

Now LHS = 0 (empty sum) and RHS = 0. Hence $P(0)$ is true

Inductive step: Let $k \in \mathbf{N}$ such that $P(k)$ is true i.e., $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

We need to show that $\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$

Now LHS = $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i \right) + (k+1)$

$$= \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)((k+1)+1)}{2} \quad = \text{RHS}$$

Hence $P(k+1)$ is true. The proof is now complete by induction.

Another Example

- **Claim**: $\forall n \in \mathbf{N},\ 3 \mid (n^3 - n)$

- **Proof** by induction on n:

Base case ($n = 0$): We need to show that $P(0)$ is true i.e., $3 \mid (0^3 - 0)$

Now $(0^3 - 0) = 0$ and since $0 = 0 \times 3$, $3 \mid 0$. Hence $P(0)$ is true

Inductive step: Let $k \in \mathbf{N}$ such that $P(k)$ is true i.e., $3 \mid (k^3 - k)$ [IH]

We need to show that $3 \mid ((k+1)^3 - (k+1))$

Now $((k+1)^3 - (k+1)) = (k^3 + 3k^2 + 3k + 1) - (k+1) = (k^3 - k) + 3(k^2 + k)$

By the IH, $3 \mid (k^3 - k)$. Also $3 \mid 3(k^2 + k)$. So $3 \mid ((k+1)^3 - (k+1))$ and hence $P(k+1)$ is true. The proof is now complete by induction.

A false proof

- **"Claim"**: $\forall n \in \mathbf{N},\ 2^n \leq n!$
- **"Proof"** by induction on n:

Base case ($n = 0$): We need to show that $P(0)$ is true i.e., $2^0 \leq 0!$

Now LHS $= 2^0 = 1 = 0!$ and hence $P(0)$ is true

Inductive step: Let $k \in \mathbf{N}$ such that $P(k)$ is true i.e., $2^k \leq k!$ [IH]

We need to show that $2^{k+1} \leq (k+1)!$

This is not true for $k = 0 \in \mathbf{N}$

Now $2^{k+1} = 2(2^k) \leq 2(k!) \leq (k+1)\times(k!) = (k+1)!$

Thus $P(k+1)$ is true, which completes the proof by induction

Fixing the previous claim and proof

- **Claim**: $\forall n \in \mathbf{N},\ n \geq 4 \rightarrow 2^n < n!$
- **Proof** by induction on n:

Base case ($n = 0$): We need to show that $0 \geq 4 \rightarrow 2^0 < 0!$

This is vacuously true, which proves the base case.

Inductive step: Let $k \in \mathbf{N}$ such that $k \geq 4 \rightarrow 2^k < k!$ [IH]
We need to show that $k+1 \geq 4 \rightarrow 2^{k+1} < (k+1)!$
Suppose $k+1 \geq 4$. There are two cases:
Case 1 ($k = 3$): Now $2^{3+1} = 2^4 = 16 < 24 = 4! = (3+1)!$
Case 2 ($k \geq 4$): Now $2^{k+1} = 2(2^k) \leq 2(k!) \leq (k+1)\times(k!) = (k+1)!$

Thus $P(k+1)$ is true, which completes the proof by induction