Another proof involving composition

- If $f : A \to B$ is onto and $g : B \to C$ is onto then $g \circ f : A \to C$ is onto
- **Proof**: Let $y \in C$. Since g is onto, $\exists z \in B$ such that $g(z) = y$

  Also since f is onto, $\exists x \in A$ such that $f(x) = z$

  Now $g \circ f(x) = g(f(x)) = g(z) = y$

  Hence, $\exists x \in A$ such that $g \circ f(x) = y$ and hence $g \circ f$ is onto
- Are the converse statements true?
  - If $g \circ f : A \to C$ is 1-to-1, is $f : A \to B$ 1-to-1 and $g : B \to C$ 1-to-1?
  - If $g \circ f : A \to C$ is onto, is $f : A \to B$ onto and $g : B \to C$ onto?
- (No, for both questions!)

Without loss of generality

- Recall that we sometimes do proofs *by cases*. Consider an example where two cases are very similar:

- Function $f : A \to B$ is increasing if $\forall x \in A, \forall y \in A, \; x < y \to f(x) < f(y)$

- **Claim**: Any increasing function is one-to-one
- **Proof**: We need to show that $\forall x \in A, \forall y \in A, \; f(x) = \mathrm{f}(y) \to x = y$
  or equivalently, $\forall x \in A, \forall y \in A, \; x \neq y \to f(x) \neq f(y)$

  Consider any x, y in A

  Case 1 ($x < y$): Since f is increasing, $f(x) < f(y)$ and hence $f(x) \neq f(y)$

  Case 2 ($y < x$): Since f is increasing, $f(y) < f(x)$ and hence $f(x) \neq f(y)$

- When the proof of the two cases is virtually identical, we can shorten this by saying: "Without loss of generality (WLOG), assume that $x < y$" and then prove just one case.

Induction

- This is a technique for proving statements of the form: $\forall n \in \mathbf{N},\ P(n)$
- A direct proof would begin "Let $n \in \mathbf{N}$. Now $P(n) = \ldots$" i.e., it is a *general* argument for why $P(n)$ is true no matter what n is
- An inductive proof has two parts:
  - — *Base case* ($n = 0$): Show that the $P(0)$ is true [usually easy]
  - — *Inductive case*: Show that $\forall k \in \mathbf{N},\ P(k) \to P(k+1)$
- To prove the inductive step, we let $k \in \mathbf{N}$ such that $P(k)$ is true, and use this to conclude that $P(k+1)$ is true
- The assumption that $P(k)$ is true is called the inductive hypothesis (IH)

Example

- **Claim**: $\forall n \in \mathrm{N}, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- **Proof** by induction on *n*:

Base case ($n = 0$): We need to show that $P(0)$ is true i.e., $\sum_{i=1}^{0} i = \frac{0(0+1)}{2}$

Now LHS = 0 (empty sum) and RHS = 0. Hence $P(0)$ is true

Inductive step: Let $k \in \mathbf{N}$ such that $P(k)$ is true i.e., $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

We need to show that $\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$

Now LHS = $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$

$$= \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)((k+1)+1)}{2} \quad = \text{RHS}$$

Hence $P(k+1)$ is true. The proof is now complete by induction.