Onto vs. One-to-one functions

- $f : A \longrightarrow B$ is onto if $\forall x \in B, \exists y \in A, f(y) = x$
- $f : A \longrightarrow B$ is one-to-one if $\forall x \in A, \forall y \in A, f(y) = f(x) \rightarrow x = y$
- *Examples*:

  $\lfloor\ \rfloor : \mathbf{R} \rightarrow \mathbf{Z}$ is onto because $\forall x \in \mathbf{Z}, \exists y = x \in \mathbf{R}, \lfloor y \rfloor = x$

  $\surd : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is one-to-one because $\forall x, y \in \mathbf{R}^+$,

  $\sqrt{x} = \sqrt{y} \rightarrow (\sqrt{x})^2 = (\sqrt{y})^2 \rightarrow x = y$

  $\surd : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is also onto because $\forall x \in \mathbf{R}^+, \exists y = x^2 \in \mathbf{R}^+, \sqrt{y} = x$

  – Thus $\surd : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is bijective

- Note that $\lfloor\ \rfloor : \mathbf{R} \rightarrow \mathbf{Z}$ is *not* one-to-one because $\lfloor 2.5 \rfloor = \lfloor 2 \rfloor$ but $2.5 \neq 2$

Composing two functions

- Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$. Then $g \circ f : A \rightarrow C$ is defined as $g \circ f(x) = g(f(x))$

  Example: Consider $\sqrt{\ }$: $\mathbf{R}^+ \rightarrow \mathbf{R}^+$ and $\lfloor\ \rfloor : \mathbf{R}^+ \rightarrow \mathbf{Z}^+$. Then $\sqrt{\ } \circ \lfloor\ \rfloor : \mathbf{R}^+ \rightarrow \mathbf{Z}^+$ is defined as $\sqrt{\ } \circ \lfloor\ \rfloor(x) = \sqrt{(\lfloor x \rfloor)}$

- Note that order of composition matters: $f \circ g \neq g \circ f$ in general!

- **Claim:** If $f : A \rightarrow B$ and $g : B \rightarrow C$ are both one-to-one, then $g \circ f : A \rightarrow C$ is also one-to-one
- **Proof**: Suppose $x \in A$, $y \in A$ such that $g \circ f(x) = g \circ f(y)$

  By definition, $g(f(x)) = g(f(y))$

  Since g is one-to-one, this means that $f(x) = f(y)$

  Since f is one-to-one, this means that $x = y$, which completes the proof