Proving two sets are equal

- In general to show that $X = Y$ we need to show two things:
  $X \subseteq Y$ and $Y \subseteq X$

- Occasionally, however, the proof follows directly by logical equivalence:

- *Example* (DeMorgan's Law): $\overline{A \cup B} = \overline{A} \cap \overline{B}$

- **Proof**:

$$
\begin{aligned}
\overline{A \cup B} &= \{ x \in U \mid x \notin A \cup B \} \\
&= \{ x \in U \mid \neg(x \in A \cup B) \} \\
&= \{ x \in U \mid \neg(x \in A \vee x \in B) \} \\
&= \{ x \in U \mid \neg(x \in A) \wedge \neg(x \in B) \} \\
&= \{ x \in U \mid (x \notin A) \wedge (x \notin B) \} \\
&= \{ x \in U \mid (x \in \overline{A}) \wedge (x \in \overline{B}) \} \\
&= \overline{A} \cap \overline{B}
\end{aligned}
$$

Try one!

- **Claim**: $A = (A - B) \cup (A \cap B)$

- **Proof**: Let $x \in A$

  There are two cases: $x \in B$ and $x \notin B$

  If $x \in B$, then $(x \in A) \wedge (x \in B)$ and hence $x \in A \cap B$
  Thus $x \in (A - B) \cup (A \cap B)$
  If $x \notin B$, then $(x \in A) \wedge (x \notin B)$ and hence $x \in (A - B)$
  Thus $x \in (A - B) \cup (A \cap B)$

  We have shown that $A \subseteq (A - B) \cup (A \cap B)$

  Now suppose $x \in (A - B) \cup (A \cap B)$, so $(x \in A - B)$ or $(x \in A \cap B)$
  In either case, $x \in A$ and hence $(A - B) \cup (A \cap B) \subseteq A$
  This completes the proof.

A proof by contradiction

- **Claim**: If $(A - B) \cup (B - A) = (A \cup B)$ then $(A \cap B) = \phi$
- **Proof**: Suppose $(A \cap B) \neq \phi$ and let $x \in A \cap B$

  Then clearly $x \in A \cup B$ and so $x \in (A - B) \cup (B - A)$

  Thus $x \in (A - B)$ or $x \in (B - A)$

  If $x \in (A - B)$ then $x \in A$ and $x \notin B$, a contradiction

  Similarly if $x \in (B - A)$ then $x \in B$ and $x \notin A$, a contradiction

  Thus we get a contradiction in every case, and hence $(A \cap B) = \phi$

A proof with power sets

- **Claim**: $A \subseteq B \leftrightarrow \mathbf{P}(A) \subseteq \mathbf{P}(B)$

- **Proof**: Suppose $A \subseteq B$ and let $S \in \mathbf{P}(A)$

  Then by definition, $S \subseteq A$
  By transitivity of $\subseteq$, $S \subseteq B$
  Hence by definition, $S \in \mathbf{P}(B)$

  Conversely, suppose $\mathbf{P}(A) \subseteq \mathbf{P}(B)$

  Since $A \in \mathbf{P}(A)$, $A \in \mathbf{P}(B)$
  Hence by definition, $A \subseteq B$

Another False Proof

- "**Claim**": $\mathbf{P}(A) \cup \mathbf{P}(B) = \mathbf{P}(A \cup B)$

- "**Proof**": Let $S \in P(A) \cup P(B)$. Then $S \in P(A)$ or $S \in P(B)$

  So $S \subseteq A$ or $S \subseteq B$

  So $S \subseteq A \cup B$

  Hence $S \in \mathbf{P}(A \cup B)$

  Let $S \in \mathbf{P}(A \cup B)$

  So $S \subseteq A \cup B$

  So $\forall x,\ x \in S \rightarrow x \in A \cup B$

  $\forall x,\ x \in S \rightarrow x \in A$ or $x \in B$

  $S \subseteq A$ or $S \subseteq B$

  So $S \in \mathbf{P}(A)$ or $S \in \mathbf{P}(B)$

  So $S \in \mathbf{P}(A) \cup \mathbf{P}(B)$