A Direct Proof with $\subseteq$

- (Transitivity of $\subseteq$): If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

- **Proof**: Let $x \in A$

  Since $A \subseteq B$, then by definition $\forall y,\ y \in A \rightarrow y \in B$

  So $x \in B$

  Again since $B \subseteq C$, then by definition $\forall z,\ z \in B \rightarrow z \in C$

  So $x \in C$

  Thus $\forall x,\ x \in A \rightarrow x \in C$

  Hence by definition, $A \subseteq C$

Another Direct Proof with $\subseteq$

- **Claim**: If $A \subseteq B$ and $C \subseteq D$ then $A \times C \subseteq B \times D$

- **Proof**: Let $(x, y) \in A \times C$

  Then by definition, $x \in A$ and $y \in C$

  Since $A \subseteq B$, $x \in B$
  and since $C \subseteq D$, $y \in D$

  Hence by definition, $(x, y) \in B \times D$

  Thus $\forall (x, y), (x, y) \in A \times C \rightarrow (x, y) \in B \times D$

  Hence by definition, $A \times C \subseteq B \times D$

A *False* Proof with $\subseteq$

- "**Claim**": If $A \times B \subseteq A \times C$ then $B \subseteq C$

- **"Proof"**: Let $y \in B$

  $\boxed{\text{Consider any } x \in A}$ ... BUT such an x may not exist (if $A = \phi$)

  Then by definition $(x, y) \in A \times B$

  So $(x, y) \in A \times C$ (since $A \times B \subseteq A \times C$)

  Then by definition $x \in A$ and $y \in C$

  Thus $\forall y,\ y \in B \rightarrow y \in C$

  Hence by definition, $B \subseteq C$

Proving two sets are equal

- In general to show that $X = Y$ we need to show two things:
  $X \subseteq Y$ and $Y \subseteq X$

- Occasionally, however, the proof follows directly by logical equivalence:

- *Example* (DeMorgan's Law): $\overline{A \cup B} = \overline{A} \cap \overline{B}$

- **Proof**: $\overline{A \cup B} = \{ x \in U \mid x \notin A \cup B \}$

$$
\begin{aligned}
&= \{ x \in U \mid \neg(x \in A \cup B) \} \\
&= \{ x \in U \mid \neg(x \in A \ \vee \ x \in B) \} \\
&= \{ x \in U \mid \neg(x \in A) \ \wedge \ \neg(x \in B) \} \\
&= \{ x \in U \mid (x \notin A) \ \wedge \ (x \notin B) \} \\
&= \{ x \in U \mid (x \in \overline{A}) \ \wedge \ (x \in \overline{B}) \} \\
&= \overline{A} \cap \overline{B}
\end{aligned}
$$