Announcements

- Quiz 1 is next week (Wednesday, 2/10)
  - Start of class, about 15 minutes long, mainly short-answer questions
  - Based on materials upto the end of *this* week
  - Skills list on the website soon
  - If you have a conflict or need extra time, please email me *ASAP*
    - kumar@illinois.edu (email me your schedule for 2/10)

Recap: Even and odd integers

- **Definition**: $n \in \mathbf{Z}$ is even if $\exists m \in \mathbf{Z},\ n = 2m$
- **Definition**: $n \in \mathbf{Z}$ is odd if $\exists m \in \mathbf{Z}, n = 2m + 1$

- For now: *Assume* that no integer is *both* even and odd
  - Try proving this yourself!

- **Definition**: $n \in \mathbf{Z}$ is a perfect square if $\exists k \in \mathbf{Z},\ n = k^2$

- "Proof" that the product of any two perfect squares is a perfect square:

Suppose n and m are perfect squares

Then, by definition, $\exists k \in \mathbf{Z},\ n = k^2$

and $\exists k \in \mathbf{Z},\ m = k^2$

So $nm = k^2.k^2 = (k^2)^2$

which is a perfect square since $k^2 \in \mathbf{Z}$

Then, by definition, $\exists k \in \mathbf{Z},\ n = k^2$

and $\exists j \in \mathbf{Z},\ m = j^2$

So $nm = k^2.j^2 = (k.j)^2$ which is a perfect square since $k.j \in \mathbf{Z}$

Rephrasing claims

- Sometimes, we are asked to prove a statement that isn't in an easy-to-prove form
  - Remember your logical equivalences!

- *Example*: There is no odd integer k such that k^2 is even

  $\equiv \neg(\ \exists k \in \mathbf{Z},\ (k \text{ is odd}) \wedge (k^2 \text{ is even})\ )$

  $\equiv \forall k \in \mathbf{Z},\ \neg(k \text{ is odd}) \vee \neg(k^2 \text{ is even})\ )$

  $\equiv \forall k \in \mathbf{Z},\ \ k \text{ is odd} \rightarrow k^2 \text{ is odd}$

- A (*very*) useful logical equivalence: $p \rightarrow q \equiv \neg q \rightarrow \neg p$ (contrapositive)

- *Example*: $\forall k \in \mathbf{Z},\ k^2 \text{ is odd} \rightarrow k \text{ is odd}$

  $\equiv \forall k \in \mathbf{Z}, \neg(k \text{ is odd}) \rightarrow \neg(k^2 \text{ is odd})$

  $\equiv \forall k \in \mathbf{Z},\ \ k \text{ is even} \rightarrow k^2 \text{ is even}$

  (easy to prove directly)

Proof by contrapositive

Proof by contrapositive

- $\forall k \in \mathbf{Z},\ 3k + 1$ is even $\rightarrow$ k is odd
  $\equiv\ \forall k \in \mathbf{Z}, \neg(k$ is odd$)\ \rightarrow\ \neg(3k + 1$ is even$)$
  $\equiv\ \forall k \in \mathbf{Z},\ k$ is even $\rightarrow\ 3k + 1$ is odd

- **Proof**: Let k be an even integer

  Then, by definition, $\exists m \in \mathbf{Z},\ k = 2m$

  Now $3k + 1$
  $= 3(2m) + 1$
  $= 2(3m) + 1$
  $= 2n + 1$, where $n = 3m \in \mathbf{Z}$

  Hence, by definition, $3k + 1$ is odd

Direct proof:

Suppose $k \in \mathbf{Z}$ and $3k + 1$ is even

Then, by definition
$\exists m \in \mathbf{Z},\ 3k + 1 = 2m$

Now $k = (2m - 1)/3$
$= ?$

Another proof by contrapositive

- $\forall a \in \mathbf{Z}, \forall b \in \mathbf{Z},\ a + b \geq 15 \ \rightarrow\ a \geq 8 \ \vee\ b \geq 8$

  $\equiv\ \forall a \in \mathbf{Z}, \forall b \in \mathbf{Z},\ \neg(a \geq 8 \ \vee\ b \geq 8) \ \rightarrow\ \neg(a + b \geq 15)$

  $\equiv\ \forall a \in \mathbf{Z}, \forall b \in \mathbf{Z},\ \neg(a \geq 8) \ \wedge\ \neg(b \geq 8) \ \rightarrow\ \neg(a + b \geq 15)$

  $\equiv\ \forall a \in \mathbf{Z}, \forall b \in \mathbf{Z},\ a < 8 \ \wedge\ b < 8 \ \rightarrow\ a + b < 15$

- **Proof**: Let a and b be integers where $a < 8$ and $b < 8$

  Hence $a \leq 7$ and $b \leq 7$

  So $a + b \ \leq\ 7 + 7 \ =\ 14 \ <\ 15$

Proof by contradiction

- In this proof-style, we prove that statement P is true as follows:

  Suppose P is *not* true.

  Then …

  … which is a contradiction.

  Hence, P must be true.

- *Example*: There is no largest even integer
- **Proof by contradiction**: Suppose there is a largest even integer k.

  Then, by definition, $\exists m \in \mathbf{Z},\ k = 2m$

  Now $k + 2$

  $= 2m + 2 = 2(m + 1)$

  which is an even integer, and $k + 2 > k$

  This a contradiction, since we assumed that k was the *largest* even integer. Hence, there is no largest even integer.

A 2,500 year old proof by contradiction

- Claim: $\sqrt{2}$ is not rational

  Note: A rational number can be written as p/q where $p, q \in \mathbf{Z}$, $q \neq 0$ and p and q have no factors in common (DEF)

- **Proof by contradiction**: Suppose $\exists r \in \mathbf{Q}$, $r = \sqrt{2}$

  Hence, $r = p/q$ where p and q satisfy all the conditions in DEF

  Now $(p/q)^2 = r^2 = (\sqrt{2})^2 = 2$

  So $p^2 = 2q^2$, and hence p^2 is even

  If p^2 is even then p is even (contrapositive of: p is odd $\rightarrow$ p^2 is odd)

  So $\exists m \in \mathbf{Z}$, $p = 2m$

  Now $2q^2 = p^2 = (2m)^2 = 4m^2$

  Hence $q^2 = 2m^2$ and hence q^2 is even

  If q^2 is even then q is even.

  So both p and q are even, which contradicts the conditions in DEF

  Thus $\sqrt{2}$ is not rational.