Announcements

- HW due TODAY, next HW released TODAY

Predicates and Variables

- A predicate is a statement that becomes a proposition when we substitute values for all variables.

- *Examples*: "$x^2 < 10$" or "My current grade in course x is y"

- We use the notation $P(x)$ to denote predicate P with variable x
  - $Q(x, y)$ denotes a predicate with two variables

- Given a predicate $P(x)$, we can ask two kinds of questions:
  - Is there a value for variable x for which $P(x)$ is true?
  - Is $P(x)$ true for all possible values of x?

- In both cases, an important point will be: what possible values can x take?

Existential Quantifier ($\exists$)

- Consider: "There is some rational number whose square is 2.25"

- In symbols, we say: $\exists x \in \mathbf{Q},\ x^2 = 2.25$
  - there exists a rational number x such that $x^2 = 2.25$
  - there exists *at least* one such rational number (actually two)
  - If we want to say "there exists exactly one x …" we say $\exists! x$

- *Example*: Let E be the set of English words. For a word $w \in E$, let reverse(w) be the word formed by reversing all the letters in w.
  - e.g., reverse(apple) = elppa

- Express these in mathematical notation:
  - there is an English word whose reverse is itself
  - there is an English word whose reverse is a different English word
- $\exists w \in E,\ w = \text{reverse}(w)$
- $\exists w \in E,\ \exists x \in E,\ (x = \text{reverse}(w) \wedge x \neq w)$

Universal Quantifier ($\forall$)

- Consider: "The square of every real number is non-negative"

- In symbols, we say: $\forall x \in \mathbf{R},\ x^2 \geq 0$
  - for all real numbers x, $x^2 \geq 0$

- *Example*: Let S be the set of English sentences. For a sentence $x \in S$, let length(x) be the total number of characters in x.
  - e.g., length(An apple fell on Newton!) = 24

- Express these in mathematical notation:
  - every English sentence has length at least three
  - every English word is contained in some English sentence

- $\forall x \in S,\ \text{length}(x) \geq 3$
- $\forall w \in E, \exists x \in S,\ x$ contains w

Proving an Existential Statement

- Consider: "There is some rational number whose square is 2.25"

- It is easy to prove that this statement is true
  - we simply need to find a rational number that works

- In general, proving "$\exists x,\ P(x)$" is easy
  - we just need to find some x for which $P(x)$ is true

- What if we want to *disprove* "$\exists x,\ P(x)$"
  - ~~"we just need to find some x for which $P(x)$ is false"~~
  - We need to prove that $P(x)$ is false *for every x*

- $\neg(\ \exists x,\ P(x)\ )\ \equiv\ \forall x,\ \neg P(x)$

- $\neg(\forall x,\ P(x)\ )\ \equiv\ \exists x,\ \neg P(x)$
  - easy to disprove a universal statement

Proving a Universal Statement

- Consider: "$\forall x \in \mathbf{Q},\ 2x \in \mathbf{Q}$"

- We can try for some particular ("random") values of $x \in \mathbf{Q}$
  - but maybe the statement doesn't hold for something we missed!

- We can try for *all* values of $x \in \mathbf{Q}$
  - but then we'll never finish our proof ($\mathbf{Q}$ is infinite)

- We can try a bunch of examples *and try to spot a pattern*
  - A proof is simply a way of capturing this pattern
  - It must be general enough (not too specific) to work for *every* $x \in \mathbf{Q}$

- Our general pattern (here) is this:
  - Every $x \in \mathbf{Q}$ looks like p/q where $p, q \in \mathbf{Z}$
  - Now $2x = 2(p/q) = (2p)/q$ where $(2p), q \in \mathbf{Z}$
  - So $2x \in \mathbf{Q}$

Proving a Universal Statement contd.

- Prove that $\forall x \in \mathbf{Q},\ 2x \in \mathbf{Q}$

- **Proof**: Consider any $x \in \mathbf{Q}$

  By definition of $\mathbf{Q}$, $x = p/q$ for some $p, q \in \mathbf{Z}$

  Now $2x = 2(p/q) = (2p)/q$ where $(2p), q \in \mathbf{Z}$

  Hence $2x \in \mathbf{Q}$ $\square$

- Notice that our proof reads like a general recipe for arguing the case for *any* given $x \in \mathbf{Q}$

- It does not rely on anything specific about the given x (e.g., whether it is an integer, or whether it is positive, etc.)
  - sometimes we need to prove "by cases"