CS 173: Discrete Structures, Spring 2010
Quiz 2 Solutions

There were two versions of the exam. This solution sheet contains the answers to all problems from both versions.

1. (4 points) Mark the following claims as "true" or "false":

   (a) There is a one-to-one function $f : \{0, 1, 2\} \to \{a, b\}$ **Solution:** False.

   (b) There is an onto function $f : \{a, b\} \to \{0, 1, 2\}$ **Solution:** False.

   (c) If $f : \mathbb{Z}^+ \to [0, 1]$ is defined as $f(n) = \frac{1}{n}$, then f is onto. **Solution:** False. For example, $\frac{2}{3}$ is in the co-domain but not in the image.

   (d) If $f : \mathbb{Z}^+ \to [0, 1]$ is defined as $f(n) = \frac{1}{n}$, then f is one-to-one. **Solution:** True. Each input maps onto a different output.

   (e) For functions f, g and h, if f is $O(g)$ and g is $O(h)$ then f is $O(h)$. **Solution:** True.

   (f) For functions f, g and h, if f is $\Omega(g)$ and g is $\Omega(h)$ then f is $\Omega(h)$. **Solution:** True.

   (g) For any $A \subseteq \mathbb{Z}$, there is a $y \in \mathbb{Z}$ such that for all $x \in A$, $x \geq y$. **Solution:** False. Such a y only exists when A has a lower bound. It won't work if A continues towards negative infinity, e.g. if A contains all the odd integers.

2. (3 points) Suppose that $f : A \to B$ is a function. Define what it means for f to be one-to-one (onto in the other version) Be specific and precise; do not use words like "unique."

   Solution: One-to-one: For every $x, y \in A$, if $f(x) = f(y)$, then $x = y$. Or: for every $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$

   Onto: For every $y \in B$, there is a $x \in A$ such that $f(x) = y$.

3. (8 points) Check all the boxes which correctly characterize each function, leaving the other boxes blank. (ALTERNATES: see end.)

   $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = 3n^3 + 7(n^2 - n)$

$O(n^3)$:	$\checkmark$	$\Omega(n^2)$:	$\checkmark$
$O(2^n)$:	$\checkmark$	$\Omega(n!)$:	

$g : \mathbb{N} \to \mathbb{N}$ such that $g(n) = 2\left(\sum_{i=0}^{n} i\right)$ — **One-to-one**: ☑ **Onto**: ☐

$h : \mathbb{N}^2 \to \mathbb{Q}$ such that $h(x, y) = \frac{x}{y+1}$ — **One-to-one**: ☐ **Onto**: ☐

$h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = \lfloor x \rfloor$ — **One-to-one**: ☐ **Onto**: ☐

$h : \mathbb{N} \to \mathbb{R}$ such that $h(n) = n \log_2(1 + n)$ — **One-to-one**: ☑ **Onto**: ☐

4. (5 points) Suppose that $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by $f(1) = 1$, $f(2) = 5$, and, for $n \geq 3$, $f(n) = 5f(n-1) - 6f(n-2)$. Fill in the following key pieces of a proof by induction showing that $f(n) = 3^n - 2^n$, for all integers $n \geq 1$. Spell out formulas explicitly; do not use vague references like "the claim" or "the formula."

Base case or cases:
Solution: This needs two base cases because the inductive step is reaching back two values.
For $n = 1$, $f(1) = 1$ and $3^n - 2^n = 3 - 2 = 1$.
For $n = 2$, $f(1) = 5$ and $3^n - 2^n = 9 - 4 = 5$.

Inductive hypothesis (i.e. what you assume at the start of the inductive step):
Solution: Suppose that $f(n) = 3^n - 2^n$ for $n = 1, \ldots, k$.

Goal of the inductive step (i.e. the conclusion you must reach at the end of the inductive step):
Solution: $f(k+1) = 3^{k+1} - 2^{k+1}$