CS 173: Discrete Structures, Fall 2009
Quiz 2 Solutions

1. (3 points) Mark the following claims as "true" or "false":

   (a) If a function $f : \mathbb{R} \to \mathbb{R}$ is increasing, then f is one-to-one.
   Solution: False. To force f to be one-to-one, you would need it to be strictly increasing. Merely increasing allows the function to have flat sections in its graph.

   (b) For any real numbers a and b, with $a > 1$ and $b > 1$, $\log_a(x)$ is $O(\log_b(x))$.
   Solution: True. Changing the base of $\log x$ changes the output by a factor that doesn't depend on x.

   (c) For any set A, $\{\emptyset\} \in \mathbb{P}(A)$.
   Solution: False. $\emptyset$ is always a member of the powerset of A, because $\emptyset$ is a subset of A. But $\{\emptyset\}$ will only be in the powerset of $\emptyset$ was a member (not just a subset) of A.

   Yup. These were tricky. Short answer theory questions are often harder than they look.

2. (8 points) Check all the boxes which correctly characterize each function, leaving the other boxes blank.

   $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = 3n^3 - 7n^2 + 37$

   $O(n^3)$: ☑ $\Omega(7n^3)$: ☑

   $O(2^n)$: ☑ $\Omega(n!)$: ☐

   $g : \mathbb{N} \to \mathbb{N}$ such that $g(n) = n!$

   One-to-one: ☐ **Onto**: ☐

   $h : \mathbb{Z} \to \mathbb{Z}$ such that $h(n) = |n|$

   One-to-one: ☐ **Onto**: ☐

   g isn't one-to-one because $0! = 1 = 1!$. That's a bit tricky.

3. (4 points) Suppose that f and g are functions whose inputs and outputs are positive real numbers. Define precisely what it means for f to be $O(g)$. Your definition must not use Θ, Ω, or slopes/derivatives.

   Solution: f is $O(g)$ if and only if there are real numbers k and C such that $f(x) \leq Cg(x)$ for every $x \geq k$.

   It's ok to put absolute values around $f(x)$ and $g(x)$, though it's not necessary given the conditions I put on the two functions. It also doesn't matter whether you used $\leq$ vs $<$, or $\geq$ vs. $>$.

4. (5 points) Suppose you are proving that $\sum_{k=1}^{n} k(k+3) = \frac{n(n+1)(n+5)}{3}$ for all positive integers n. Supply the following key pieces of a proof by induction on n. Spell out the details explicitly; don't use vague references like "the claim" or "the formula."

   Base case or cases: **Solution:** $n = 1$. Then $\sum_{k=1}^{n} k(k+3) = 1 \cdot 4 = 4$ and $\frac{n(n+1)(n+5)}{3} = \frac{1 \cdot 2 \cdot 6)}{3} = 4$.

   Inductive hypothesis (i.e. what you assume at the start of the inductive step): **Solution:** $\sum_{k=1}^{p} k(k+3) = \frac{p(p+1)(p+5)}{3}$ for some positive integer p.

   Goal of the inductive step (i.e. the conclusion you must reach at the end of the inductive step): **Solution:** $\sum_{k=1}^{p+1} k(k+3) = \frac{(p+1)(p+2)(p+6)}{3}$