CS 173, Fall 2009
Midterm 1 Solutions

Problem 1: Short answer (12 points)

State whether each of the following claims is true or false. Justification/work is not required, but may increase partial credit if your short answer is wrong.

(a) $-6 \mid -3$

Solution: False. The magnitude of -6 is larger, so there's no way you could have an integer k such that $-6k = -3$.

(b) For every integer x, if x is a negative prime number then $x^2 < 0$.

Solution: True. It's vacuously true because no value of x will make the hypothesis true.

(c) $\lceil x \rceil < \lfloor x + 1 \rfloor$ for any real number x?

Solution: False. Suppose $x = 3.1$. Then $\lceil x \rceil = 4 = \lfloor x + 1 \rfloor$.

(d) For any set A, $A \in \mathbb{P}(A)$.

Solution: True, because A is a subset of itself.

(e) For any positive integers p and q, if $\text{lcm}(p, q) = pq$, then p and q are relatively prime.

Solution: True. $\text{lcm}(p, q) = \frac{pq}{\gcd(p,q)}$ and the definition of "relatively prime" is $\gcd(p, q) = 1$.

(f) $\mathbb{P}(A \cup B) = \mathbb{P}(A) \cup \mathbb{P}(B)$, for any sets A and B.

Solution: False. $A \cup B$ is often larger because it can contain "mixed" subsets, with some elements from A and some elements from B.

Problem 2: Calculation (11 points)

Calculate the values of the following expressions. Your answer to (e) must be in closed form, i.e. not using a summation. Recall that $\mathbb{P}(A)$ is the power set of A.

(a) $|\{\{2, 2, 3\}, \{7\}, \{19, 23, 23\}\}| =$

Solution: 3. You only count the top-level elements.

(b) $\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x \leq y \leq 3\} =$

Solution: $\{(0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

(c) $\{1, 2\} \times \mathbb{P}(\{3\}) =$

Solution: $\{(1, \emptyset), (2, \emptyset), (1, \{3\}), (2, \{3\})\}$

(d) Suppose that A is a p-element set and B is a q-element set. Then $|A \times \mathbb{P}(B)| =$

Solution: $p2^q$

(e) (3 points) $\sum_{k=2}^{p}(k + 2) =$

Solution: $2(p-1) + \sum_{k=2}^{p} k = (2p-2) - 1 + \sum_{k=1}^{p} k = 2p - 3 + \frac{p(p+1)}{2}$

Problem 3: Longer answers (7 points)

(a) (3 points) Suppose that $f : A \to B$ is a function. Briefly explain the difference between the co-domain of f and the image of f.

Solution: The co-domain is the declared set that potential output values must come from (B in this case). The image is the set of values you actually get if you apply the function f to all the objects in A. [There's many ways you might have expressed this idea. Slightly informal is ok as long as you clearly convey understanding of the distinction.]

(b) (4 points) Write pseudocode for a function that computes the gcd of two positive integers using the Euclidean algorithm. It is not critical to use precisely the same coding conventions as in lecture/Rosen, but it must be pseudocode (not e.g. C or Java code) and easy to understand.

Solution:

```
procedure gcd(a,b: positive integers)
x := a
y := b
while y != 0
   begin
   r := x mod y
   x := y
   y := r
   end
return x
```

Recursive variants are also fine.

Problem 4: Logic and proof structure (6 points)

(a) (3 points) Give the negation of the following statement moving the negatives (e.g. "not") so that they are on individual predicates (e.g. "my bike has two wheels"). It's ok to give your answer in logic shorthand but, if you do, make sure that you have clearly indicated definitions for your predicate or set symbols (e.g. "$R(x)$ means x is red").

> For every city c, there is a road r such that if c got money from House Bill 3141, then c fixed the potholes in r.

Solution: There is a city c, such that for every road r, c got money from House Bill 3141 and c fixed the potholes in r.

(b) (3 points) Disprove the following claim by giving a concrete counterexample and briefly explaining why it is a counterexample.

> For any integers a, b, and any positive integer p, if $(a \bmod p) < (b \bmod p)$, then $((a+3) \bmod p) < ((b+3) \bmod p)$.

Solution: Suppose $a = 2$, $b = 3$, and $p = 6$. Then $(a \bmod p) = 2 < 3 = (b \bmod p)$ but $((a+3) \bmod p) = 5 > 0 = ((b+3) \bmod p)$. Many choices for the three values would work. However, it's not ok to give a general argument in place of a specific set of values.

Problem 5: Writing a proof (7 points)

Recall the following definition: given any positive integer m, the integers a and b are *congruent modulo m* (in shorthand: $a \equiv b \pmod{m}$) if and only if there is an integer k such that $a = b + km$.

Prove that, for any integers p, q, s, and t and any positive integer m,

> If $p \equiv q \pmod{m}$ and $s \equiv t \pmod{m}$, then $ps \equiv qt \pmod{m}$.

Prove this directly using the above definition, together with high school algebra. Do not use other facts about modular arithmetic proved in class or in the book. Make sure your steps are in logical order.

Solution: Let p, q, s, and t be integers and m a positive integer and suppose that $p \equiv q \pmod{m}$ and $s \equiv t \pmod{m}$.

Since $p \equiv q \pmod{m}$, $p = q + km$, where k is an integer (by the definition above). Similarly, since $s \equiv t \pmod{m}$, $s = t + jm$, where j is an integer. So

$$ps = (q + km)(t + jm) = qt + (kt + qj + kjm)m$$

Since k, j, q, t, and m are all integers, $(kt + qj + kjm)$ is an integer. Let's call it b.

Then $ps = qt + bm$, which implies that $pq \equiv qt \pmod{m}$.

Problem 6: Proof by contradiction (7 points)

Prove the following claim using proof by contradiction. You **must** use this proof method and the following definition of "multiple": n is a multiple of m if and only if $n = km$ for some integer k. Work directly from this definition using algebra; do not use other facts from class or the textbook about divisibility. You can, however, assume that we know which small integers are multiples of one another, e.g. 13 is not a multiple of 4.

> For any natural numbers p and q, if p and q are multiples of 3, then the equation $pn + qm = 5$ has no integer solution. (That is, it is not possible to find integers n and m which make this equation true.)

Solution: Suppose not. That is suppose that p and q are natural numbers, p and q are multiples of 3, and the equation $pn + qm = 5$ has an integer solution.

Since p and q are multiples of 3, $p = 3s$ and $q = 3t$ for some integers s and t. So $pn + qm = 3sn + 3tm = 3(sn + tm)$. This means that $pn + qm$ is a multiple of 3. But we assumed that $pn + qm = 5$ and 5 is not a multiple of 3. So we have a contradiction.

Therefore, our claim must have been true.