CS 173, Fall 2009
Final Exam, 17 December 2009

Fill in your name, netid, and discussion section time below. Also write your name or netid on the last page (which sometimes gets pulled off).

NAME:

NETID:

DISCUSSION DAY/TIME:

Problem	1	2	3	4	5
Possible	8	13	11	12	8
Score					
Problem	6	7	8	9	10
Possible	10	9	9	10	10
Score					

Total out of 100 points

INSTRUCTIONS (read carefully)

- There are 10 problems, each on a single page. Make sure you have a complete exam.
- The point value of each problem is indicated next to the problem.
- It is wise to skim all problems and point values first, to best plan your time.
- Points may be deducted for solutions which are correct but excessively complicated, hard to understand, or poorly explained.
- Brief explanations and/or showing work may increase partial credit for buggy answers.
- We expect most people to finish the exam in 2 hours, but you can take up to the full 3 hours.
- Turn in your exam at the front. Show your ID to the proctors.
- This is a closed book exam. No notes or electronic devices of any kind are allowed.
- Except where explicitly indicated, it isn't necessary to simplify or calculate out complex constant expressions such as $(0.7)^3(0.3)^5$, $\frac{0.15}{3.75}$, $7!$, $\binom{10}{7}$, and the like.
- Do all work in the space provided, using the backs of sheets if necessary. See the proctor if you need more paper.
- Please bring any apparent bugs to the attention of the proctors.
- After the exam is over, discuss its contents with other students **only** after verifying that they have also taken the exam (e.g. they aren't about to take a conflict exam).

Problem 1: Multiple choice (8 points)

Check the appropriate box for each statement. (One box per statement.) If you change your answer, make it very clear when you've meant to uncheck a box.

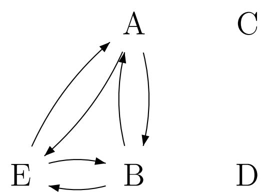
There are exactly 7 Platonic solids (polyhedra with uniform vertex and face degrees).	**True** ☐	**False** ☐
If a function from $\mathbb{R}$ to $\mathbb{R}$ is strictly increasing, it must be one-to-one.	**True** ☐	**False** ☐
There exist mathematical functions that cannot be computed by any C program.	**True** ☐	**False** ☐
If G is a connected planar graph with v vertices, e edges, and f faces, then $e - f + v = 2$.	**True** ☐	**False** ☐
There are relations which are both symmetric and antisymmetric.	**True** ☐	**False** ☐
For any set A, then $\emptyset \in A$.	**True** ☐	**False** ☐

The worst-case running time of the Towers of Hanoi solver $\Theta(n \log n)$ ☐ $\Theta(n^2)$ ☐ $\Theta(2^n)$ ☐ $\Theta(n!)$ ☐

$T(1) = c$
$T(n) = 3T(n/3) + d$ $\theta(n)$: ☐ $\theta(n^2)$: ☐ $\theta(n \log n)$: ☐ $\theta(2^n)$: ☐

Problem 2: Checkbox (13 points)

Check all boxes which correctly characterize each relation or function, leaving the other boxes blank. (If you change your answer, make it very clear when you've meant to uncheck a box.)



Reflexive: ☐ **Irreflexive**: ☐

Symmetric: ☐ **Antisymmetric**: ☐

Transitive: ☐

$f : \mathbb{N} \to \mathbb{N}$
such that
$f(x) = 7x^2 + 3$

One-to-one: ☐ **Onto**: ☐

$f : \mathbb{R} \to \mathbb{R}$
such that
$g(n) = 3n^3 - 7n + 1$

$O(n^2)$: ☐ $O(n^3)$: ☐ $O(n^4)$: ☐

$\Omega(n^2)$: ☐ $\Omega(n^3)$: ☐ $\Omega(n^4)$: ☐

Problem 3: Short answer (11 points)

(a) (4 points) Give a closed form for the summation $\sum_{k=3}^{n}(2^n + n)$.

(b) (3 points) State (in words) the contrapositive of the sentence

> If it is snowing and it is Sunday, then it is below freezing and it is cloudy.

(c) (4 points) In the expansion of the polynomial $(3x + 7y)^{17}$, what is the coefficient of the x^5y^{12} term?

Problem 4: Short answer (12 points)

(a) (4 points) Rajiv wants to give 15 cans of vegetables to the food drive. The grocery store sells cans of peas, corn, chickpeas, and pumpkin. In how many ways can Rajiv choose his set of cans?

(b) (4 points) Suppose that $f : A \to B$ is a function. Define what it means for f to be one-to-one. Make your definition precise and unambiguous. Do not use words such as "distinct" or "unique."

(c) (4 points) Define the relation $\ll$ on $\mathbb{R}^3$ such that $(x, y) \ll (p, q)$ if and only if $x^2 + y^2 \leq p^2 + q^2$. Use concrete counter-examples to show that $\ll$ is neither symmetric nor anti-symmetric.

Problem 5: Counting (8 points)

A domino consists of two squares ("ends"). Each end may be blank or may contain between one and n spots. A domino may have the same number of spots on both ends, or different numbers of spots on the two ends. A double-n domino set contains exactly one of each possible dot combination, where the order of the two ends doesn't matter. For example, a double-two domino set would contain: $(0,0)$, $(1,0)$, $(2,0)$, $(1,1)$, $(1,2)$, $(2,2)$.

(a) (4 points) How many dominoes are there in a double-nine set? Show your work or explain why your formula is correct.

(b) (4 points) Give a general formula for the number of dominoes in a double-n set, explaining why your formula is correct.

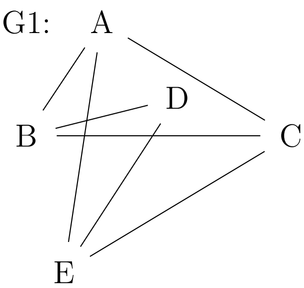
Problem 6: Short proofs (10 points)

(a) (5 points) Prove that $3n^2 + 7$ is $O(n^2)$. Prove this directly from the definition of what it means for a function f to be $O(g)$ (where g is another function), being careful to justify your algebraic steps and put them into logical order.

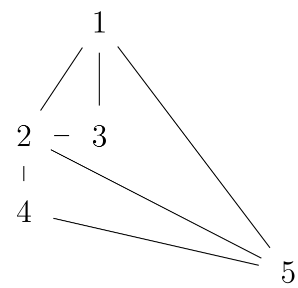
(b) (5 points) Recall that $a \mid b$ if $b = ka$ for some integer k. Prove that, for all integers a, b, and c, if $c \mid a$ and $c \mid b$, then $c \mid 3b - a^2$.

Problem 7: Trees and graphs (9 points)

G1:



G2:



(a) (3 points) Prove that the graph G1 (above) is planar.

(b) (3 points) Prove that the graphs G1 and G2 (above) are not isomorphic.

(c) (3 points) For which values of n does the graph $K_{2,n}$ have an Euler circuit? Briefly justify your answer.

Problem 8: Algorithm analysis (9 points)

The procedure Foo takes as input an array of n positive integers $a_1, a_2, ...a_n$ and reorders this array. Its code is as follows:

```
procedure Foo(a_1,..., a_n)
t := a_1
for k := 1 to n − 1
    begin
    a_k := a_{k+1}
    end
a_n := t
Foo(a_1,..., a_{n−1})
```

(a) (3 points) If Foo is run on the input array $4, 13, 20, 5, 8, 10$, what does the reordered array look like after Foo is finished? Describe in one sentence what sort of re-ordering Foo does.

(b) (3 points) Let $T(n)$ be the running time of Foo on an input array of length n. Express $T(n)$ as a recurrence relation. Be sure to include an initial condition.

(c) (3 points) Give a big-theta bound on $T(n)$. Briefly explain why your answer is correct and/or show work (e.g. some steps from unrolling your recurrence).

Problem 9: Writing a proof (10 points)

Suppose that $\sim$ is an equivalence relation on a set A. Recall the definition of an equivalence class

$$[x] = \{p \in A \mid x \sim p\}$$

Using this definition and the fact that $\sim$ is an equivalence relation (i.e. reflexive, symmetric, and transitive) and being careful to justify each step in your proofs, prove that

(a) (5 points) For any x and y in A, if $[x] \cap [y] \neq \emptyset$, then xRy.

(b) (5 points) For any x and y in A, if xRy, then $[x] \subseteq [y]$.

Problem 10: Induction (10 points)

Recall the definition of the Fibonacci numbers:

$$F_0 = 0,\ F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2}.$$

The Lucas numbers are defined by

$$L_0 = 2,\ L_1 = 1, \text{ and } L_n = L_{n-1} + L_{n-2}.$$

Use induction to prove that $L_n = F_{n-1} + F_{n+1}$ for every integer $n \geq 1$.

Base case or cases:

Inductive hypothesis:

Main work of the inductive step:

Conclusion of your inductive step: