CS 173, Spring 2010
Midterm 1 Solutions

Problem 1: Short answer (12 points)

State whether each of the following claims is **TRUE** or **FALSE**. Justification/work is not required, but may increase partial credit if your short answer is wrong.

(a) For all integers p and q, if $p \mid q$ then q must be positive. **Solution:** False.

(b) For all prime numbers p, there are exactly two natural numbers q such that $q \mid p$. **Solution:** True.

(c) $8 \equiv 11 \pmod 3$ **Solution:** True.

(d) There is a set A such the cardinality of $\mathbb{P}(A)$ is less than two. **Solution:** True.

(e) For all positive integers p and q, $\gcd(p,q) \leq \text{lcm}(p,q)$. **Solution:** True.

(f) For all sets A and B, $\mathbb{P}(A \cap B) \subseteq \mathbb{P}(A \cup B)$. **Solution:** True.

Problem 2: Set theory calculation (10 points)

Suppose that $A = \{4,5,6\}$ and $B = \{2,7,8,11,13\}$. Calculate the values of the following expressions (recall that $\mathbb{P}(X)$ is the power set of X). Explicitly list the contents of non-empty sets.

(a) $A \cup \{4, \{5,6\}, 11\} =$ **Solution:** $\{4,5,6,11,\{5,6\}\}$

(b) Cardinality of $\mathbb{P}(A \times B) =$ **Solution:** $2^{3 \cdot 5} = 2^{15}$

(c) $\mathbb{P}(A) \cap \mathbb{P}(B) =$ **Solution:** $\{\emptyset\}$

(d) $\{p \in B \mid p^2 \in A\} =$ **Solution:** $\{2\}$

(e) $A \times \{a, \emptyset\} =$ **Solution:** $\{(4,a),(5,a),(6,a),(4,\emptyset),(5,\emptyset),(6,\emptyset)\}$

Problem 3: Longer answers (8 points)

(a) (3 points) Trace the execution of the Euclidean algorithm as it computes gcd(1012, 299). Clearly indicate the return value.

Solution:

x	y	r
1012	299	115
299	115	69
115	69	46
69	46	23
46	23	0

Return value is 23.

(b) (2 points) Is the following claim considered **TRUE** or **FALSE** in mathematics? Briefly explain why your answer is correct.

For all integers m, if $m^2 + 2 < 2$, then $\sqrt{2}$ is rational.

Solution: Notice that the a minus sign in the original printed exam was corrected to a + sign when the exam was given. This statement is (vacuously) true, because the hypothesis is always false.

(c) (3 points) Give a closed-form expression for the following summation, showing your work. (You do not need to simplify the closed-form.)

$$\sum_{k=2}^{n+1}(7k + 2^{-k}) =$$

Solution:

$$\begin{aligned}
\sum_{k=2}^{n+1}(7k + 2^{-k}) &= (\sum_{k=0}^{n+1}(7k + 2^{-k})) - (0+1) - (7 + \frac{1}{2}) \\
&= (7(\sum_{k=0}^{n+1} k)) + ((\sum_{k=0}^{n+1} 2^{-k})) - 8.5 \\
&= 7\frac{(n+1)(n+2)}{2} + (2 - \frac{1}{2^{n+1}}) - 8.5 \\
&= 7\frac{(n+1)(n+2)}{2} - \frac{1}{2^{n+1}} - 6.5
\end{aligned}$$

Problem 4: Notation and definitions (9 points)

(a) Suppose that a and b are integers, and k a positive integer. Define what it means for a and b to be congruent mod k ($a \equiv b \pmod{k}$). Base your definition directly on algebra. Do not use the divides relation or the mod numerical operator.

Solution: $a - b = mk$ for some integer m.

(b) Negate the following statement, moving all negations (e.g. "not") so they are on individual predicates. Your final answer should be in English, like the given statement.

For every integer n, there is some integer p, such that $pn > 0$.

Solution: There is an integer n such that for every integer p, $pn \leq 0$.

(c) We defined a mod b (where b is positive) using the "division algorithm" theorem. If a mod $b = r$, then part of this definition/theorem stipulates that $a = bq + r$ for some integer q. What is the second condition that r must satisfy?

Solution: $0 \leq r < b$

Problem 5: Proof structure (7 points)

(a) (3 points) Consider the following fragment of a proof involving variables that are natural numbers (k not zero):

...We now know that $k = \frac{pq}{k}$. So $k^2 = pq$. Thus pq is a *perfect square*, and hence p must be equal to q.

Describe the flaw in this reasoning and give a counterexample illustrating how it can fail.

Solution: If pq is a perfect square, then it's equal to $m \cdot m$ for some integer m. But p and q aren't prime, then m might not be p or q. For example, if $p = 9$ and $q = 16$, then pq is the square of 12.

(b) (4 points) Prove the following claim **using proof by contradiction**: $\sqrt{\sqrt{2}}$ is not a rational number. You can use the fact (proved in class) that $\sqrt{2}$ is not rational.

Solution: Suppose not. That is, suppose that $\sqrt{\sqrt{2}}$ is a rational number. Then $\sqrt{\sqrt{2}} = \frac{p}{q}$ for some integers p and q (q not zero). If we square both sides, we then find that $\sqrt{2} = \frac{p^2}{q^2}$. But p^2 and q^2 must be integers, since p and q are integers. So this means that $\sqrt{2}$ is rational, which contradicts the fact (proved in class) that it's not rational.

Problem 6: Writing a proof (7 points)

Recall that a real number m is rational if $m = \frac{p}{q}$ for some integers p and q, where $q \neq 0$. Using this definition and high school algebra, prove the following claim:

For any real numbers p and q, if p and q are rational, then $q + 3p$ is rational.

Solution: Suppose that p and q are rational numbers. Then $p = \frac{a}{b}$ and $q = \frac{m}{n}$ where a, b, m, n are integers (b and n non-zero).

Then $q + 3p = \frac{m}{n} + 3\frac{a}{b} = \frac{bm+3an}{bn}$.

Notice that $bm + 3an$ and bn are both integers, since they are made by multiplying and adding variables known to be integers. And bn can't be zero, because b and n are both non-zero. So $q + 3p = \frac{bm+3an}{bn}$ implies that $q + 3p$ is rational.

Problem 7: Proof by contrapositive (7 points)

Use proof by contrapositive to prove the following claim.

For all real numbers x, if $x^2 - x - 6 > 0$, then $x > 3$ or $x < -2$.

Begin by explicitly stating the contrapositive of the claim:

Solution: For all real numbers x, if $x \leq 3$ and $x \geq -2$, then $x^2 - x - 6 \leq 0$.

The rest of your proof:

Solution: Suppose that x is a real number such that $x \leq 3$ and $x \geq -2$. Since $x \leq 3$, $x - 3 \leq 0$. Since $x \geq -2$, $x + 2 \geq 0$. Then $(x - 3)(x + 2) \leq 0$ because the product of a negative number and a positive number is negative and the product will be zero if either input is zero. But $x^2 - x - 6 = (x - 3)(x + 2)$. So $(x - 3)(x + 2) \leq 0$.