CS 173, Final Exam
7 May 2010

Fill in your name, netid, and discussion section time below. Also write your name or netid on the last page (which sometimes gets pulled off).

NAME:

NETID:

DISCUSSION DAY/TIME:

Problem	1	2	3	4	5
Possible	8	13	12	11	12
Score					
Problem	6	7	8	9	10
Possible	10	7	7	10	10
Score					

Total out of 100 points

INSTRUCTIONS (read carefully)

- There are 10 problems, each on a single page. Make sure you have a complete exam.
- The point value of each problem is indicated next to the problem.
- It is wise to skim all problems and point values first, to best plan your time.
- Points may be deducted for solutions which are correct but excessively complicated, hard to understand, or poorly explained.
- Brief explanations and/or showing work may increase partial credit for buggy answers.
- We expect most people to finish the exam in 2 hours, but you can take up to the full 3 hours.
- Turn in your exam at the front. Show your ID to the proctors.
- This is a closed book exam. No notes or electronic devices of any kind are allowed.
- Except where explicitly indicated, it isn't necessary to simplify or calculate out complex constant expressions such as $(0.7)^3(0.3)^5$, $\frac{0.15}{3.75}$, $7!$, $\binom{10}{7}$, and the like.
- Do all work in the space provided, using the backs of sheets if necessary. See the proctor if you need more paper.
- Please bring any apparent bugs to the attention of the proctors.
- After the exam is over, discuss its contents with other students **only** after verifying that they have also taken the exam (e.g. they aren't about to take a conflict exam).

Problem 1: Multiple choice (8 points)

Check the appropriate box for each statement. (One box per statement.) If you change your answer, make it very clear when you've meant to uncheck a box.

In 1759, Leonhard Euler proved that any planar graph can be colored with four colors. **True** ☐ **False** ☐

For any sets A and B, if $A \cap B = \emptyset$, then $\mathbb{P}(A) \cap \mathbb{P}(B) = \emptyset$. **True** ☐ **False** ☐

In a binary tree, it is possible for a leaf node to be the root. **True** ☐ **False** ☐

Suppose graph with 12 vertices is colored with exactly 5 colors. By the pigeonhole principle, each color appears on at least two vertices. **True** ☐ **False** ☐

The set of rationals $\mathbb{Q}$ has the same cardinality as the set of integers $\mathbb{Z}$. **True** ☐ **False** ☐

For any positive integers n and k, $\binom{n}{k} = \binom{n}{n-k}$. **True** ☐ **False** ☐

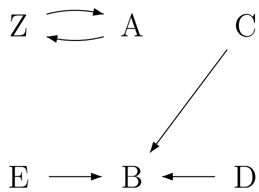
For any integer k, $\gcd(0, k) = 0$. **True** ☐ **False** ☐

$T(1) = d$
$T(n) = 3T(n-1) + d$

$\theta(n^3)$: ☐ $\theta(n \log n)$: ☐ $\theta(3^n)$: ☐ $\theta(n!)$: ☐

Problem 2: Checkbox (13 points)

Check all boxes which correctly characterize each relation or function, leaving the other boxes blank. (If you change your answer, make it very clear when you've meant to uncheck a box.)



Reflexive: ☐ **Irreflexive**: ☐

Symmetric: ☐ **Antisymmetric**: ☐

Transitive: ☐

$f : \mathbb{N} \to \mathbb{N}$
such that
$f(x) = \lfloor \log_2(1 + 2x) \rfloor$

One-to-one: ☐ **Onto**: ☐

$g : \mathbb{R} \to \mathbb{R}$
such that
$g(n) = n^2 + 3n \log n - 37$

$O(3n \log n)$: ☐ $O(n^2)$: ☐ $O(2^n)$: ☐

$\Omega(3n \log n)$: ☐ $\Omega(n^2)$: ☐ $\Omega(2^n)$: ☐

Problem 3: Short answer (12 points)

(a) (4 points) Let $A = \{a, b\}$ and $B = \{9, 12\}$. Write out the members of $A \times \mathbb{P}(B)$.

(b) (4 points) If p, q, and r are natural numbers, how many solutions does the equation $p + q + r = 17$ have?

(c) (4 points) For $n \geq 1$, let S_n be the set of binary strings of length n in which two zeros are never consecutive. For example, `010` $\in S_3$ and `111` $\in S_3$ but `11` $\notin S_3$ and `001` $\notin S_3$. If $n \geq 3$, any member of S_n has one of two forms:

- `1` followed by a member of S_{n-1} (e.g. `1110` is `1` followed by `110`), or
- `01` followed by a member of S_{n-2} (e.g. `0110` is `01` followed by `10`).

Let $T_n = |S_n|$. Express T_n as a recurrence, including a base case or cases. (**Do not** find a closed form for the recurrence.)

Problem 4: Short answer (11 points)

(a) (4 points) Suppose that f and g are functions whose domain and co-domain are the positive reals. Using precise mathematical language, define what it means for f to be $O(g)$. (Do not use related notions such as Ω.)

(b) (3 points) State (in words) the negation of the sentence: For every martian M, if M is green, then M is tall and ticklish.

(c) (4 points) Define the set of real intervals $J = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x < y\}$. The relation R on J is defined by $(x, y)R(p, q)$ if and only if $x \leq p \leq y$ and $y \leq q$. Prove that R is not transitive.

Problem 5: Counting and relations (12 points)

Consider a non-empty set A containing n objects. Answer each of the following questions, briefly explaining your answer and/or showing work.

(a) (4 points) How many relations on A are *both* symmetric and reflexive?

(b) (4 points) How many relations on A are symmetric? *Hint*: Express your answer in terms of your solution to part (a).

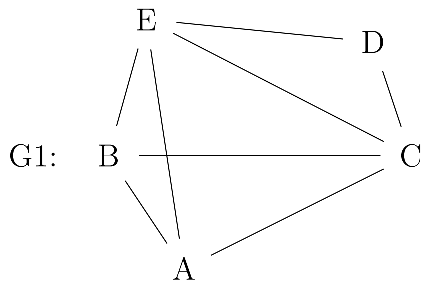
(c) (4 points) How many *equivalence relations* on A have exactly two equivalence classes? *Hint*: How many ways are there to partition the set A into two nonempty subsets?

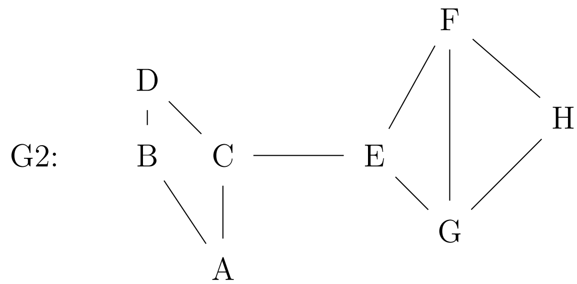
Problem 6: Short proofs (10 points)

(a) (5 points) Suppose that $f : \mathbb{N}^2 \to \mathbb{N}^2$ is defined by $f(x, y) = (5y, x^2 + y)$. Prove that f is one-to-one, being careful to justify your algebraic steps and put them into logical order.

(b) (5 points) Suppose that R is a relation on a set A. Prove that if R is irreflexive and transitive, then R must be antisymmetric.

Problem 7: Graphs (7 points)





(a) (3 points) What is the chromatic number of the graph G1 (above)? Briefly justify your answer.

In graph G2 (above), the edge $\{C, E\}$ is called a "cut edge," because removing the edge $\{C, E\}$ (without changing the vertices), would divide the graph into two disconnected pieces. Recall that each of these pieces is called a "connected component."

(b) (1 point) Suppose that a simple connected planar graph has v vertices, e edges, and f faces. How does deleting a cut edge change the values of v, e, and f?

(c) (3 points) Euler's formula for connected planar graphs (i.e. a single connected component) states that $v - e + f = 2$. State the generalization of Euler's formula for planar graphs with k connected components (where $k \geq 1$). Briefly justify your answer. *Hint*: Use part (b).

Problem 8: Algorithm analysis (7 points)

The procedure maxSubsequence takes as input an array of positive integers $a_1, a_2, \ldots, a_n$ (where $n \geq 1$) and finds the length of the *longest* strictly increasing contiguous subsequence. E.g., maxSubsequence(1, 3, 2, 4, 7) = 3 where the longest contiguous subsequence is 2, 4, 7.

```
1  maxSubsequence(a_1, ..., a_n : array of positive integers)
2          maxlen := 1
3          i := 1
4          while (i ≤ n)
5                  s := i
6                  while (i < n and a_i < a_{i+1})
7                          i := i + 1
8                  if (i − s + 1 > maxlen)
9                          maxlen := i − s + 1
10                 i := i + 1
11         return maxlen
```

(a) (3 points) Let $q(n)$ be the number of times i is increased (lines 7 and 10) on an input of length n. Give a big-Θ bound on $q(n)$. Briefly explain why your answer is correct.

(b) (2 points) In the worst case, how many times will line 5 be executed on an input of length n? Clearly describe what sorts of inputs cause this worst-case behavior.

(c) (2 points) What is the big-Θ running time of maxSubsequence?

Problem 9: Writing a proof (10 points)

Recall the definition of congruence mod k (where k is a positive integer): if p and q are integers, $p \equiv q \pmod{k}$ if and only iff $k \mid (p - q)$ for some integer k. Using this definition and high school algebra, prove that:

> For any integers a, b, c, d, and positive integer k, if $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ then $a^2 + c \equiv b^2 + d \pmod{k}$.

Do not use other facts about modular arithmetic that you may remember from class. Be sure to use good mathematical style, with the steps in logical order and appropriate justifications.

Write your netID, in case this page gets pulled off:

Problem 10: Induction (10 points)

Recall the definition of the Fibonacci numbers:

$$F_0 = 0,\ F_1 = 1, \text{ and } \forall n \geq 2,\ \ F_n = F_{n-1} + F_{n-2}.$$

Use induction to prove that $F_n \leq 2^{n-2} - 1$ for every integer $n \geq 4$.

Base case or cases:

Inductive hypothesis: [Be specific]

Main work of the inductive step:

Conclusion of your inductive step: