Grammar Trees

Define a grammar G_1 by $S \to aSbS \mid SaS \mid ab \mid a$. where S is the only start symbol and the terminal symbols are a and b. Prove that a tree generated by G_1 has at least as many nodes labeled a as nodes labeled b.

Solution:

Proof by induction on the height h of the tree.

Base Cases: The grammar G_1 has no trees with height 0 but there are two trees with height 1. Those trees are generated by the rules $S \to ab$ and $S \to a$. These trees both clearly have more a nodes than b nodes so the base cases hold.

IH: Assume that for all trees defined by the grammar G_1 with height h k have at least as many a nodes as b nodes.

Now consider a Tree T defined by the grammar G_1 with height $h = k$.

There are two cases to consider that are not covered by the base cases.

Case 1: A tree with the first expansion from the root being the rule $S \to aSbS$. In this case the root will have two children that are leaves one with a label a and the other b. It will also have two children that are sub-trees T_1, T_2 each of which have S nodes as their roots so are trees of the grammar G_1 and since they are below the root of the main tree have heights $< k$. Thus by the inductive hypothesis have at least as many nodes labeled a than labeled b. So a groups of nodes have at least as many a nodes as b nodes. Thus in this case a tree generated by G_1 has at least as many nodes labeled a as nodes labeled b

Case 2: A tree with the first expansion from the root being the rule $S \to SaS$. In this case the root will have one child that is a leaf with a label a. It will also have two children that are sub-trees T_1, T_2 each of which have S nodes as their roots so are trees of the grammar G_1 and since they are below the root of the main tree have heights $< k$. Thus by the inductive hypothesis have at least as many nodes labeled a than labeled b. So a groups of nodes have at least as many a nodes as b nodes. Thus in this case a tree generated by G_1 has at least as many nodes labeled a as nodes labeled b

So in all cases a tree generated by G_1 has at least as many nodes labeled a as nodes labeled b. $\square$

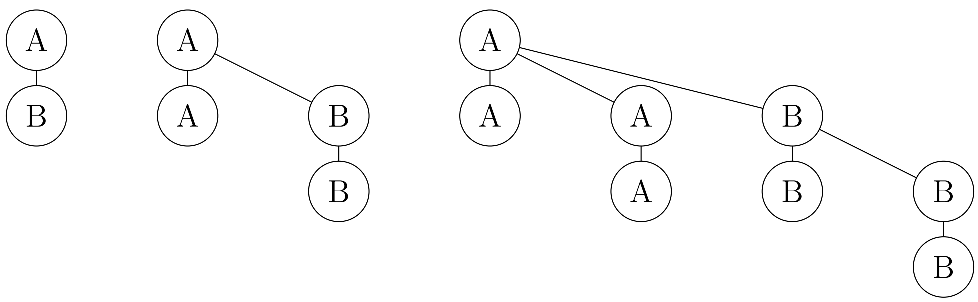
M-nomial Trees

A **m-nomial** of order m is defined recursively as follows:

(1) A single root node is a m-nomial tree of order 0.

(2) A m-nomial tree of order m consists of two m-nomial trees of order $m-1$, with the root of the second connected as the rightmost child of the root of the first.

The following picture shows the m-nomial trees of order 1, 2, and 3. The labels on the nodes show how the larger tree is divided into two lower-order subtrees.



Use induction on the order of the tree to prove that a m-nomial tree of order m has 2^m nodes.

Solution:

Proof by induction on the order m of a m-nomial tree.
Base Case: A m-nomial tree of order $m = 0$ has by definition 1 node and $2^0 = 1$.
IH: Assume that a m-nomial tree of order $m < k$ has 2^m nodes.
Consider a m-nomial tree of order k. By definition, this tree is composed of two m-nomial trees of order $k-1$. Since $k-1 < k$ by the inductive hypothesis, these each have 2^{k-1} nodes. So the total tree has $2 \cdot 2^{k-1} = 2^k$ nodes. Which shows that a tree of order k has 2^k nodes. So with the base cases and induction we have proven that all m-nomial trees of order m have 2^m nodes. $\square$

Parity Trees

A parity tree is a full binary tree with each node colored orange or blue such that:

1. If v is a leaf node, then v is colored orange.
2. If v has two children of the same color, then v is colored blue.
3. If v has two children of different colors, then v is colored orange.

Prove by induction that every parity tree has the parity property that if the root is colored orange, then it has an odd number of leaves; and if the root is colored blue, then it has an even number of leaves.

Solution:

Proof by induction on the height h of the tree.

Base Cases:

The shortest parity tree is height 0 and has one node which is a leaf, so is orange. Since there is one leaf, the number of leaves is odd and the root is orange, which matches the parity property.

IH: Assume that all parity trees with height $h < k$ have the parity property.

Consider a parity tree of height k. There are two cases.

1. Case orange: If the root is orange, the tree must have two children with roots of different colors. one must be orange and the other must be blue. Since both subtrees are of height less than k so we can apply the inductive hypothesis. This means that one has an odd number of leaves and the other has an even number. Since all leaves are in the subtrees and an odd plus an even is an odd we can clearly see that the number of leaves is odd which is what is required for the parity property.
2. Case blue: If the root is blue, the tree must have two subtrees with root nodes of the same color. These subtrees are of height less than k so we can apply the inductive hypothesis. If the roots are blue then they both have an even number of leaves and if the roots are orange then they both have an odd number of leaves and since either an even plus an even or an odd plus an odd is an even number we can see the tree has an even number of leaves which matches the parity property.

In both cases we have a tree with the parity property. Thus in both the inductive step and bases cases we have shown that we have the parity property so all parity trees have the the parity property.