14th of October, 2025

1. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined recursively below.

$$f(n) := \begin{cases} 2 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 3f(n-1) - 2f(n-2) & \text{if } n \geqslant 2 \end{cases}$$

   Prove that $f(n) = 2^n + 1$ for all $n \in \mathbb{N}$.

2. Consider the function $g : \mathbb{N}_+ \to \mathbb{N}_+$ defined recursively below.

$$g(1) \coloneqq 1$$
$$g(z) \coloneqq 2g(z-1) + 3 \quad \text{for all } z \in \mathbb{N} - \{0, 1\}$$

Find a closed form for g and prove your conjecture by induction.

3. When we analyse the `binary_search` algorithm, we might describe the amount of computational work it takes to search through a list with the recursive function $T : \mathbb{N}_+ \to \mathbb{N}_+$ given below.

$$T(n) := \begin{cases} 4 & \text{if } n = 1 \\ T(n/2) + 4 & \text{if } n \geqslant 2 \end{cases}$$

   Find a closed form for T and prove your conjecture by induction.

4. When we analyse `merge_sort`, we might describe the amount of computational work it takes to sort a list with the recursive function $T : \mathbb{N}_+ \to \mathbb{N}_+$ given below.

$$T(n) := \begin{cases} 0 & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n \geqslant 2 \end{cases}$$

   Find a closed form for T and prove your conjecture by induction.