Given a predicate $\varphi(\cdot)$ of one free variable, the schema of (strong) mathematical induction is as follows.

$$(\forall n \in \mathbb{N})\big(\varphi(n)\big) \;\Leftrightarrow\; \varphi(0) \wedge (\forall k \in \mathbb{N})\Big(\big(\forall \ell \in \mathbb{N}\big)\big(l \leqslant k \Rightarrow \varphi(\ell)\big) \Rightarrow \varphi(k+1)\Big).$$

1. Prove that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

Proof. We will prove the claim by induction on $\mathbb{N}$.

Basis Step:
Observe that $\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2^{0+1} - 1$.

`NTS:` $\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$

Inductive Step:
Let $k \in \mathbb{N}$ and assume that $(\forall \ell \in \mathbb{N})\Big(\ell \leqslant k \Rightarrow \sum_{i=0}^{\ell} 2^i = 2^{\ell+1} - 1\Big)$. We can now bear witness to the following derivation.

`IH:` $(\forall j \in \mathbb{N})\Big(j \leqslant k \Rightarrow \sum_{i=0}^{\ell} 2^i = 2^{\ell+1} - 1\Big)$

`NTS:` $\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$

$$\begin{aligned}
\sum_{i=0}^{k+1} 2^i &= \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} && \text{by def. of summation} \\
&= 2^{k+1} - 1 + 2^{k+1} && \text{by the \textit{inductive hypothesis}} \\
&= 2 \cdot 2^{k+1} - 1 && \text{because } a + a = 2a \text{ for all } a \in \mathbb{R} \\
&= 2^{k+2} - 1 && \text{because } a \cdot a^b = a^{b+1} \text{ for all } a, b \in \mathbb{R}
\end{aligned}$$

Thus, we have $\sum_{i=0}^{k+1} 2^i = 2^{k+1}$ as desired.

We therefore conclude $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \in \mathbb{N}$. Q.E.D.

2. Prove that $\sum_{i=1}^{n}(i+1)2^i = n \cdot 2^{n+1}$ for all $n \in \mathbb{N}_+$.

Proof. We will prove the claim by induction on $\mathbb{N} - \{0\}$.

Basis Step:
Observe that $\sum_{i=1}^{1}(i+1)2^i = (1+1)2^1 = 4 = 1 \cdot 2^{1+1}$.

NTS: $\sum_{i=1}^{1}(i+1)2^i = 1 \cdot 2^{1+1}$

Inductive Step:
Let $k \in \mathbb{N}_+$ and assume, for all $\ell \in \mathbb{N}$, that $\sum_{i=1}^{\ell}(i+1)2^i = \ell \cdot 2^{\ell+1}$ whenever $\ell \leqslant k$. We can now observe the following.

IH: $(\forall \ell \in \mathbb{N}_+)\Big(\ell \leqslant k \Rightarrow \sum_{i=1}^{\ell}(i+1)2^i = \ell 2^{\ell+1} - 1\Big)$

NTS: $\sum_{i=1}^{k+1}(i+1)2^i = (k+1)2^{k+2}$

$$\begin{aligned}
\sum_{i=1}^{k+1}(i+1)2^i &= \left(\sum_{i=1}^{k}(i+1)2^i\right) + ((k+1)+1)2^{k+1} && \text{by def. of summation} \\
&= k \cdot 2^{k+1} + (k+2)2^{k+1} && \text{by the } \textit{inductive hypothesis} \\
&= k \cdot 2^{k+1} + k \cdot 2^{k+1} + 2 \cdot 2^{k+1} && \text{by distributivity of } \cdot \text{ over } + \\
&= k \cdot 2 \cdot 2^{k+1} + 2 \cdot 2^{k+1} && \text{because } a + a = 2a \text{ for all } a \in \mathbb{R} \\
&= k \cdot 2^{k+2} + 2^{k+2} && \text{because } a \cdot a^b = a^{b+1} \text{ for all } a, b \in \mathbb{R} \\
&= (k+1)2^{k+2} && \text{by factoring out } 2^{k+2}
\end{aligned}$$

Thus, we have $\sum_{i=1}^{k+1}(i+1)2^i = (k+1)2^{k+2}$ as desired.

Therefore, we can conclude $\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}$ for all $n \in \mathbb{N}_+$.

Q.E.D.

3. Prove that $2+3n<2^n$ for all natural numbers $n>3$.

Proof. We will prove the claim by induction on $\mathbb{N}-\{0,1,2,3\}$.

Basis Step:
Observe that $2+3\cdot 4=2+12=14<16=2^4$.

NTS: $2+3\cdot 4<2^4$

Inductive Step:
Let $k\in\mathbb{N}$ such that $k>3$. Assume, for all $\ell\in\mathbb{N}$ such that $\ell>3$, that $2+3\ell<2^\ell$. Notice, since $k>3$, that $2^k>2^3=8>3$ because the exponential is a strictly increasing function. We can now simply observe.

IH: $(\forall\ell\in\mathbb{N})\Big(3<\ell\leqslant k\Rightarrow 2+3\ell<2^\ell\Big)$

NTS: $2+3(k+1)<2^{k+1}$

$$\begin{aligned}2+3(k+1)&=(2+3k)+3 && \text{by distributing}\\ &<2^k+3 && \text{by the \textit{inductive hypothesis}}\\ &<2^k+2^k && \text{because we proved } 3<2^k\\ &=2^{k+1} && \text{because } a^b+a^c=a^{b+c} \text{ for all } a,b,c\in\mathbb{R}\end{aligned}$$

We thus know $2+3(k+1)<2^{k+1}$ as desired.

Therefore, we can conclude $2+3n<2^n$ for all $n\in\mathbb{N}-\{0,1,2,3\}$.

Q.E.D.

4. Prove that $\displaystyle\sum_{i=0}^{n}\binom{n}{i} = 2^n$ for all $n \in \mathbb{N}$.

Proof. We will prove the claim by induction on $\mathbb{N}$. Recall Pascal's identity, which states that $\binom{a+1}{b+1} = \binom{a}{b} + \binom{a}{b+1}$ for all $a, b \in \mathbb{N}$.

Basis Step:

Observe that $\sum_{i=0}^{0}\binom{0}{i} = \binom{0}{0} = 1 = 2^0$. NTS: $\sum_{i=0}^{0}\binom{0}{i} = 2^0$

Inductive Step:

Let $k \in \mathbb{N}$ and suppose $(\forall \ell \in \mathbb{N})\left(\sum_{i=0}^{\ell}\binom{\ell}{i} = 2^\ell\right)$. Now, watch closely. IH: $(\forall \ell \in \mathbb{N})\left(\sum_{i=0}^{\ell}\binom{\ell}{i} = 2^\ell\right)$

NTS: $\sum_{i=0}^{k+1}\binom{k+1}{i} = 2^{k+1}$

$$
\begin{aligned}
\sum_{i=0}^{k+1}\binom{k+1}{i} &= \sum_{i=0}^{k}\binom{k+1}{i} + \binom{k+1}{k+1} && \text{by extracting the last term from the sum}\\
&= \sum_{i=1}^{k}\binom{k+1}{i} + \binom{k+1}{0} + \binom{k+1}{k+1} && \text{by extracting the first term from the sum}\\
&= \sum_{i=0}^{k-1}\binom{k+1}{i+1} + \binom{k+1}{0} + \binom{k+1}{k+1} && \text{by down-shifting indices of summation}\\
&= \sum_{i=0}^{k-1}\binom{k}{i} + \sum_{i=0}^{k-1}\binom{k}{i+1} + \binom{k+1}{0} + \binom{k+1}{k+1} && \text{by } \binom{a+1}{b+1} = \binom{a}{b} + \binom{a}{b+1} \text{ for all } a, b \in \mathbb{N}\\
&= \sum_{i=0}^{k-1}\binom{k}{i} + \sum_{i=1}^{k}\binom{k}{i} + \binom{k+1}{0} + \binom{k+1}{k+1} && \text{by up-shifting indices of summation}\\
&= \sum_{i=0}^{k-1}\binom{k}{i} + \sum_{i=1}^{k}\binom{k}{i} + \binom{k}{0} + \binom{k+1}{k+1} && \text{because } \binom{k+1}{0} = 1 = \binom{k}{0}\\
&= \sum_{i=0}^{k-1}\binom{k}{i} + \sum_{i=1}^{k}\binom{k}{i} + \binom{k}{0} + \binom{k}{k} && \text{because } \binom{k+1}{k+1} = 1 = \binom{k}{k}\\
&= \sum_{i=0}^{k}\binom{k}{i} + \sum_{i=1}^{k}\binom{k}{i} + \binom{k}{0} && \text{bringing the last term into the sum}\\
&= \sum_{i=0}^{k}\binom{k}{i} + \sum_{i=0}^{k}\binom{k}{i} && \text{bringing the first term into the sum}\\
&= 2^k + 2^k && \text{by the \textit{inductive hypothesis}}\\
&= 2^{k+1} && \text{because } a^b + a^c = a^{b+c} \text{ for all } a, b, c \in \mathbb{R}
\end{aligned}
$$

This demonstrates that $\sum_{i=0}^{k+1}\binom{k+1}{i} = 2^{k+1}$ as desired.

We therefore conclude that $\displaystyle\sum_{i=0}^{n}\binom{n}{i} = 2^n$ for all $n \in \mathbb{N}$. Q.E.D.