7^{th} of October, 2025

Given a predicate $\varphi(\cdot)$ of one free variable, the schema of (strong) mathematical induction is as follows.

$$(\forall n \in \mathbb{N})\big(\varphi(n)\big) \iff \varphi(0) \wedge (\forall k \in \mathbb{N})\Big(\big(\forall \ell \in \mathbb{N}\big)\big(l \leqslant k \Rightarrow \varphi(\ell)\big) \Rightarrow \varphi(k+1)\Big).$$

1. Prove that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

2. Prove that $\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}$ for all $n \in \mathbb{N}_+$.

3. Prove that $2 + 3n < 2^n$ for all natural numbers $n > 3$.

4. Prove that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ for all $n \in \mathbb{N}$.