1. Consider the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{Z}$ given by $f(x, y) := \left\lfloor \frac{x}{y+1} \right\rfloor$.

   (a) We will show that f is *not* one-to-one.

   Proof. Consider the ordered pairs $(1, 0) \in \mathbb{R} \times \mathbb{N}$ and $(1.5, 0) \in \mathbb{R} \times \mathbb{N}$ and observe.

$$f(1,0) = \left\lfloor \frac{1}{0+1} \right\rfloor = \lfloor 1 \rfloor = 1 = \lfloor 1.5 \rfloor = \left\lfloor \frac{1.5}{0+1} \right\rfloor = f(1.5, 0)$$

   However, $(1, 0) \neq (1.5, 0)$ because $1 \neq 1.5$. Therefore, f is not one-to-one. Q.E.D.

   (b) We will show that f is onto.

   Proof. Let $y \in \mathbb{Z}$ and note that $(y, 0) \in \mathbb{R} \times \mathbb{N}$ because $\mathbb{Z} \subseteq \mathbb{R}$. Observe that $f(y, 0) = \lfloor y/0+1 \rfloor = \lfloor y \rfloor$. Since y is an integer, we can see that $\lfloor y \rfloor = y$, implying $f(y, 0) = y$. Therefore, f is onto. Q.E.D.

2. Let $f : \mathbb{N} \to \mathbb{N}$ be a one-to-one function.
   Consider the function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ given by $g(x, y) := \big(f(x) - y, f(x) + y\big)$.

   (a) We will show that g is one-to-one.

   Proof. Let $(a, b) \in \mathbb{N} \times \mathbb{N}$ and $(x, y) \in \mathbb{N} \times \mathbb{N}$. Assume that $g(a, b) = g(x, y)$. By definition, this implies $\big(f(a) - b, f(a) + b\big) = \big(f(x) - y, f(x) + y\big)$, yielding the following two equalities.

$$f(a) - b = f(x) - y$$
$$f(a) + b = f(x) + y$$

   We then have $\big(f(a) - b\big) + \big(f(a) + b\big) = \big(f(x) - y\big) + \big(f(x) + y\big)$, from which we derive the following.

$$\begin{aligned} \big(f(a) - b\big) + \big(f(a) + b\big) = \big(f(x) - y\big) + \big(f(x) + y\big) &\Rightarrow 2f(a) + (b - b) = 2f(x) + (y - y) \\ &\Rightarrow 2f(a) = 2f(x) \\ &\Rightarrow f(a) = f(x) \end{aligned}$$

   Because f is one-to-one, we now know $a = x$. Since $f(a) + b = f(x) + y$, we can use the intermediate result $f(a) = f(x)$ to get $b = y$. We therefore conclude $(a, b) = (x, y)$, showing f is one-to-one. Q.E.D.

   (b) We will show that g is *not* onto.

   Proof. Consider the ordered pair $(0, -1) \in \mathbb{Z} \times \mathbb{Z}$. Since the codomain of f is $\mathbb{N}$, we know that, $\forall n \in \mathbb{N}, f(n) \geqslant 0$ and that $\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, f(n) + m \geqslant 0$. We then can *not* have $f(n) + m = -1$ for any $n, m \in \mathbb{N}$, showing $\forall n, m \in \mathbb{N}, g(n, m) = (f(n) - m, f(n) + m) \neq (0, -1)$. Thus, g is not onto. Q.E.D.

3. Consider the function $h : \mathbb{Z} \to \mathbb{N}$ given by $h(x) = \begin{cases} 2x & \text{if } x \geqslant 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$.

(a) We will show h is one-to-one.

Proof. Let $a, b \in \mathbb{N}$ and assume $h(a) = h(b)$. We will take four cases.

Case 1:
Assume $a \geqslant 0$ and $b \geqslant 0$. Then, by definition, $h(a) = 2a = 2b = h(b)$, so that $a = b$.

Case 2:
Assume $a \geqslant 0$ and $b < 0$. Then, $h(a) = 2a$ and $h(b) = -2b - 1$. Notice that $h(a)$ is even, while $h(b)$ is odd; this implies $h(a) \neq h(b)$. However, this contradicts our earlier knowledge that $h(a) = h(b)$. Having derived a contradiction, the assumption that $a \geqslant 0$ and $b < 0$ must be false, making this case irrelevant to our proof.

Case 3:
Assume $a < 0$ and $b \geqslant 0$. Then, $h(a) = -2a - 1$ and $h(b) = 2b$. Notice that $h(a)$ is odd, while $h(b)$ is even; this implies $h(a) \neq h(b)$. However, this contradicts our earlier knowledge that $h(a) = h(b)$. Having derived a contradiction, the assumption that $a < 0$ and $b \geqslant 0$ must be false, making this case irrelevant to our proof.

Case 4:
Assume $a < 0$ and $b < 0$. Then, $h(a) = -2a - 1 = -2b - 1 = h(b)$, implying $-2a = -2b$, so that $a = b$.

We therefore know that $a = b$, allowing us to conclude that h is one-to-one. Q.E.D.

(b) We will show h is onto.

Proof. Let $y \in \mathbb{N}$. We take two cases.

Case 1:
Assume y is even, so that $y = 2k$ for some $k \in \mathbb{Z}$ by definition. Since $y \in \mathbb{N}$, we know $2k \geqslant 0$, implying $k \geqslant 0$. We can then observe that $f(k) = 2k = y$ by definition.

Case 2:
Assume y is odd, so that $y = 2k + 1$ for some $k \in \mathbb{Z}$. Since $y \in \mathbb{N}$, we can make the following derivation.

$$\begin{aligned} y \geqslant 0 &\Rightarrow 2k + 1 \geqslant 0 \\ &\Rightarrow 2k > 0 \\ &\Rightarrow k > 0 \\ &\Rightarrow -k < 0 \\ &\Rightarrow -k - 1 < k < 0 \end{aligned}$$

We then have $f(-k-1) = -2(-k-1) - 1 = 2k + 2 - 1 = 2k + 1 = y$ by definition.

Thus, in both cases, we have found $\exists z \in \mathbb{Z}, f(z) = y$. Therefore, f is onto. Q.E.D.