## 0.1 Introduction

An interesting result from number theory is that every positive rational number can be written as a *simple finite continued fraction* of the form given below.

$$q = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_\ell}}}} = [a_0; a_1, a_2, \ldots, a_\ell]$$

$[a_0; a_1, a_2, \ldots, a_\ell]$ is a compact modern notation for expressing (finite) simple continued fractions. With this compact notation, we say that $\ell$ is the *length* of the continued fraction and that $a_0, a_1, a_2, \ldots, a_\ell$ are its *coefficients*. The first coefficient $a_0$ is the *integral coefficient*, and the rest are *fractional coefficients*.

In the above representation, each positive rational number $q \in \mathbb{Q}_+$ has an associated finite sequence of *coefficients* $a_0, a_1, a_2, \ldots, a_\ell$ where $a_0 \in \mathbb{N}$ and $a_1, a_2, \ldots, a_\ell \in \mathbb{N}_+$.

$$\frac{2}{1} = 1 + \frac{1}{1} \qquad [1; 1]$$

$$\frac{2}{3} = 0 + \cfrac{1}{1 + \cfrac{1}{1+1}} \qquad [0; 1, 1, 1]$$

$$\frac{415}{93} = 4 + \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{6+1}}} \qquad [4; 2, 6, 6, 1]$$

We can *normalize*[1] the representation $[a_0; a_1, a_2, \ldots, a_\ell]$ of a positive rational number by additionally restricting $\ell > 0$ and $a_\ell = 1$. Taking this as inspiration, we recursively define a bijection $f : \mathbb{N}_+ \to \mathbb{Q}_+$ based on alternating addition and reciprocation.

$$f(1) := 1$$

$$f(2n) := 1 + f(n)$$

$$f(2n+1) := \frac{1}{1 + f(n)}$$

[1] An interesting thing to note is that every positive rational number has *exactly two* representations as a finite simple continued fraction: one in which the last coefficient (after $a_0$) is 1, and one in which the last coefficient (after $a_0$, if it exists) is nonzero. We can therefore *normalize* the continued fraction representation by requiring that there exist at least one coefficient after $a_0$ and that the *last* coefficient $a_n$ is 1. With this *normalization*, each positive rational number will have a *unique* representation as a continued fraction.

The output of $f$ on the first few positive natural numbers is shown below.

$$f(1) = 1 \qquad = {}^1/_1$$

$$f(2) = 1 + f(1) = \frac{1}{1} + \frac{1}{1} = {}^2/_1$$

$$f(3) = \frac{1}{1 + f(1)} = \frac{1}{1+1} = {}^1/_2$$

$$f(4) = 1 + f(2) = \frac{1}{1} + \frac{2}{1} = {}^3/_1$$

$$f(5) = \frac{1}{1 + f(2)} = \frac{1}{1+2} = {}^1/_3$$

$$f(6) = 1 + f(3) = 1 + \frac{1}{2} = {}^3/_2$$

$$f(7) = \frac{1}{1 + f(3)} = \frac{1}{1 + \frac{1}{2}} = {}^2/_3$$

$$f(8) = 1 + f(4) = 1 + \frac{3}{1} = {}^4/_1$$

# 1 Injectivity

Let's prove that $f$ is one-to-one by induction on its input. To be clear, we are proving that $(\forall m \in \mathbb{N}_+)(f(n) = f(m) \Rightarrow n = m)$ for all $n \in \mathbb{N}_+$ by induction on $n$.

*Proof.* First, observe that $f(x) > 0$ for all $x \in \mathbb{N}_+$ because the codomain of $f$ is $\mathbb{Q}_+$. This implies $f(2x) = 1 + f(x) > 1$ and $f(2x+1) = \frac{1}{1+f(x)} < 1$ for all $x \in \mathbb{N}_+$.

*Basis Step:*
Let $m \in \mathbb{N}_+$ and assume $f(1) = f(m)$, so that $f(m) = 1$. Towards a contradiction, suppose $m \neq 1$. If $m$ is even, then $f(m) = 1 + f({}^{m}/_{2}) > 1$,[2] contradicting the fact that $f(m) = 1$. However, if $m$ is odd, then $f(m) = \frac{1}{1+f(\frac{m-1}{2})} < 1$,[3] again contradicting $f(m) = 1$. ↯
Therefore, we know $m = 1$.

[2] $\because \forall x \in \mathbb{N}_+, f(x) > 0.$

[3] $\because \forall x \in \mathbb{N}_+, f(x) > 0.$

*Inductive Step:*
Let $k \in \mathbb{N}_+$. Assume, for all $i, j \in \mathbb{N}_+$, that if $i < k$ and $f(i) = f(j)$, then $i = j$.[4]
Now, let $m \in \mathbb{N}_+$ and assume $f(k) = f(m)$.

[4] This statement is our inductive hypothesis.

*Case 1:*
Suppose $k$ is even, so that $f(k) = 1 + f(\frac{k}{2})$. Towards a contradiction, suppose $m$ is odd, so that $f(m) = \frac{1}{1+f(\frac{m-1}{2})}$ by definition.[5] This implies $1 + f(\frac{k}{2}) = \frac{1}{1+f(\frac{m-1}{2})}$, which we can rearrange to produce the following expression.

$$\left(1 + f\left(\frac{k}{2}\right)\right)\left(1 + f\left(\frac{m-1}{2}\right)\right) = 1$$

However, $\left(1 + f(\frac{k}{2})\right)\left(1 + f(\frac{m-1}{2})\right) > 1$ because $f$ is strictly positive. ↯

[5] Since $m$ is odd, we know $m = 2a + 1$ for some $a \in \mathbb{Z}$. This implies $\frac{m-1}{2} = a \in \mathbb{Z}$.

Therefore, $m$ is even. This implies $f(m) = 1 + f(\frac{m}{2})$. Since $1 + f(\frac{k}{2}) = 1 + f(\frac{m}{2})$, we then have $f(\frac{k}{2}) = f(\frac{m}{2})$. Applying the *inductive hypothesis,* we get $\frac{k}{2} = \frac{m}{2}$, so that $k = m$ as desired.

*Case 2:*
Suppose $k$ is odd, so $f(k) = \frac{1}{1+f(\frac{k-1}{2})}$. Towards a contradiction, suppose that $m$ is even, so that $f(m) = 1 + f(\frac{m}{2})$. Through similar reasoning as in the previous case, we can see the following.

$$\left(1 + f\left(\frac{k-1}{2}\right)\right)\left(1 + f\left(\frac{m}{2}\right)\right) = 1$$

Just as before, we know $\left(1 + f(\frac{k-1}{2})\right)\left(1 + f(\frac{m}{2})\right) > 1$ because $f$ is strictly positive. ↯

Therefore, $m$ is odd, implying $f(m) = 1/\left(1 + f(\frac{m-1}{2})\right)$. Recalling that $f(k) = f(m)$, we obtain $1 + f(\frac{k-1}{2}) = 1 + f(\frac{m-1}{2})$, implying that $f(\frac{k-1}{2}) = f(\frac{m-1}{2})$. Applying the *inductive hypothesis* then produces $\frac{k-1}{2} = \frac{m-1}{2}$, so that $k = m$ as desired.

Therefore, in either case, we can see that $k = m$, finishing our inductive step, and concluding our proof that $f$ is injective. QED

# 2 Surjectivity

Recall that each positive rational number $q \in \mathbb{Q}_+$ can be written as a continued fraction of the form $a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_\ell}}}$ where $a_0 \in \mathbb{N}$ and $a_1, \ldots, a_\ell, \ell \in \mathbb{N}_+$ and $a_\ell = 1$.

I would like to thank the absolute chads of section `CL1` and `CL2` of `CS173` (whose names will be added here as soon as I track them down) for their contributions to a special case of an earlier version of this result, and I would like to thank Grace Sun for her insight that *significantly* simplified this proof.

To prove $f$ is onto, we will perform induction on $\sum_{i=0}^{n} a_i$, the *sum of the coefficients.* In order to have a hope of accomplishing this properly, we need to be clear about precisely what it is that we are proving. We will show, for every $n \in \mathbb{N}_+$, for every $q \in \mathbb{Q}_+$, if $q = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_\ell}}}$ such that $\sum_{i=0}^{\ell} a_i = n$, then $(\exists x \in \mathbb{N}_+)(f(x) = q)$.

*Proof.* We proceed by induction on the sum of the coefficients in the continued fraction representation of a given positive rational number.

*Basis Step:*
Let $q \in \mathbb{Q}_+$ and suppose $q = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_\ell}}}$ such that $\sum_{i=0}^{\ell} a_i = 1$, where $a_0 \in \mathbb{N}$ and $a_1, \ldots, a_\ell, \ell \in \mathbb{N}_+$ and $a_\ell = 1$. Because there must be at least two coefficients in the continued fraction, and since $a_i \geqslant 1$ for each $i \geqslant 1$, we can use the fact that $\sum_{i=0}^{\ell} a_i = 1$ to deduce that $\ell = 2$ and thus $a_0 = 0$ and $a_1 = 1$. As a result, we know $q = 1$, and we can therefore conclude $f(1) = 1 = q$.

*Inductive Step:*
Let $k \in \mathbb{N}_+$ and suppose, for any $j \in \mathbb{N}_+$ such that $j \leqslant k$, for any $q \in \mathbb{Q}_+$ for which we have $q = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_\ell}}}$ with $a_0 \in \mathbb{N}$ and $a_1, \ldots, a_\ell, \ell \in \mathbb{N}_+$ and $a_\ell = 1$, that if $\sum_{i=0}^{\ell} a_i = j$, then there exists $x \in \mathbb{N}_+$ such that $f(x) = q$.[6]

[6] This statement is our inductive hypothesis.

Now that the inductive hypothesis is established, let $q$ be a positive rational number such that $q = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_\ell}}}}$ where $a_0 \in \mathbb{N}$ and $a_1, a_2, \ldots, a_\ell, \ell \in \mathbb{N}_+$ and $a_\ell = 1$ with the property that $\sum_{i=0}^{\ell} a_i = k + 1$. We now define the positive rational $\tilde{q} \in \mathbb{Q}_+$ as follows.

$$\tilde{q} = (a_1 - 1) + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_\ell}}}$$

Since $(a_1 - 1) + a_2 + \ldots + a_\ell < a_0 + a_1 + a_2 + \ldots + a_\ell = k + 1$, we know that there exists $\tilde{x} \in \mathbb{N}+$ such that $f(\tilde{x}) = \tilde{q}$ by the *inductive hypothesis.* Observe.

$$f(2\tilde{x} + 1) = \frac{1}{f(\tilde{x})} = \frac{1}{1 + \tilde{q}} = \frac{1}{1 + (a_1 - 1) + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_\ell}}}} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_\ell}}}}$$

The second branch of the recursive definition of $f$ takes the reciprocal of $1 + \tilde{q}$

Observe again.

$$f(2^{a_0}(2\tilde{x} + 1)) = a_0 + f(2\tilde{x} + 1) = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_\ell}}}} = q$$

Every time we take the first recursive branch in the definition of $f$ by multiplying the input by 2, we add 1 to the output. Doing this $a_0$ times, we multiply the input by $2^{a_0}$, adding $a_0$ to the output.

We have therefore found that $2^{a_0}(2\tilde{x} + 1)$ is a preimage for $q$ under $f$ as desired.

This concludes the proof by induction that $f$ is surjective. QED