1. Prove that $\forall x(\varnothing \subseteq x)$.

   Proof: Let x be an arbitrary set. Assume, towards a contradiction, that $\varnothing \nsubseteq x$.
   By definition, this means there exists some z such that $z \in \varnothing$ and $z \notin x$.
   However, since $\varnothing$ is *empty*, we know $\forall w(w \notin \varnothing)$; in particular, this implies $z \notin \varnothing$. ↯
   Therefore, we conclude $\varnothing \subseteq x$. Q.E.D.

2. Show that $\forall x(x \in \mathbb{P}(x))$.

   Proof: Let x be an arbitrary set. Assume, towards a contradiction, that $x \notin \mathbb{P}(x)$.
   This means $x \nsubseteq x$ by definition, which implies there exists z such that $z \in x$ and $z \notin x$. ↯
   Therefore, we conclude $x \in \mathbb{P}(x)$. Q.E.D.

3. Prove that no natural number can be both even and odd at the same time.

   Proof: Let $n \in \mathbb{N}$ and assume, towards a contradiction, that n is even and n is odd.
   Since n is even, we know there exists $k \in \mathbb{Z}$ such that $n = 2k$.
   Since n is odd, we know there exists $\ell \in \mathbb{Z}$ such that $n = 2\ell + 1$.
   It then follows that $2k = 2\ell + 1$, so that $2(k - \ell) = 1$. We now have three cases.

   Case 1: Suppose $k - j < 0$. Then, $1 = 2(k - j) < 0$. However, we know that $1 \geqslant 0$. ↯

   Case 2: Suppose $k - j = 0$. Then, $1 = 2(k - j) = 0$. However, we know that $1 \neq 0$. ↯

   Case 3: Suppose $k - j > 0$. Since $k - j \in \mathbb{Z}$, this implies $k - j \geqslant 1$, so that $1 = 2(k - j) \geqslant 2$.
   However, we know that $1 < 2$. ↯

   Since we've reached a contradiction in each case, and these cases are exhaustive, we can conclude that n can not be both even and odd simultaneously. Q.E.D.

4. Show that there are no positive integers x and y that solve $x^2 - y^2 = 10$.

   Proof: Towards a contradiction, suppose there exist $x, y \in \mathbb{N}_+$ such that $x^2 - y^2 = 10$. The following equalities then follow.

   $$(x + y)(x - y) = x^2 - y^2 = 10 = 2 \cdot 5$$

   This implies $x + y$ and $x - y$ are divisors of 10. Since $x + y > x - y$, this means $x + y \in \{5, 10\}$ in particular. We now have two cases.

   Case 1: Suppose that $x + y = 10$, so that $x - y = 1$. We can then see the following is true.

   $$2x = (x + y) + (x - y) = 10 + 1 = 11$$

   This implies that 11 is even. However, 11 is odd, contradicting our previous result. ↯

   Case 2: Suppose that $x + y = 5$, so that $x - y = 2$. Then, we have the following.

   $$2x = (x + y) + (x - y) = 5 + 2 = 7$$

   This implies that 7 is even. However, 7 is odd, contradicting our previous result. ↯

   Having reached contradictions in each case, and knowing that our cases are exhaustive, we can therefore conclude that there are no positive integer solutions to the given equation. Q.E.D.

5. Prove that every graph with 2 or more nodes contains two distinct nodes with the same degree.

   Proof: Let G be a graph on n nodes, where $n \in \mathbb{N}$ and $n \geqslant 2$. Define V as the set of all vertices in G. Recall $\deg(v)$ is the number of edges incident on v for each $v \in V$ and define D as follows.

   $$D := \{\deg(v) \mid v \in V\}$$

   Now, consider the function $f : V \to D$ given by $f(v) := \deg(v)$. We would like to apply the Pigeonhole Principle to f; in order to do this, we will need to show that $|V| > |D|$.

   First, we know that $|V| = n$. Second, notice that $(\forall v \in V)(0 \leqslant \deg(v) \leqslant n-1)$, which implies $D \subseteq \{0, ..., n-1\}$. This means $|D| \leqslant n$. We can now take two cases.

   Case 1: Suppose there are no nodes in G of degree 0. Then, $D \subseteq \{1, ..., n-1\}$, so $|D| \leqslant n-1$.

   Case 2: Suppose there exists a node $x \in V$ such that $\deg(x) = 0$. Towards a contradiction, assume that there exists a node $y \in V$ such that $\deg(y) = n-1$. Since there are n nodes in G, there are $n-1$ nodes in G distinct from y; therefore, y must be connected by an edge to each of the other nodes in G. In particular, this implies y is connected to x, which means there is an edge incident on x, which implies $\deg(x) > 0$. However, we know that $\deg(x) = 0$. ↯ Therefore, there are no nodes of degree $n-1$ in G. From this, $D \subseteq \{0, ..., n-2\}$, so that can derive $|D| \leqslant n-1$.

   From our case analysis, we now know $|D| \leqslant n-1$. Thus, by the Pigeonhole Principle, we can conclude that f is not a one-to-one function. This means, by definition, that there exist vertices $v, w \in V$ such that $v \neq w$ and $\deg(v) = f(v) = f(w) = \deg(w)$, as desired. Q.E.D.

6. For every set X, show $|X| < |\mathbb{P}(X)|$ by proving that there is no onto function from X to $\mathbb{P}(X)$.

   Proof: Let X be a set and assume, towards a contradiction, that there exists $f : X \to \mathbb{P}(X)$ such that f is onto. Consider the set Δ defined as follows.

   $$\Delta := \{x \in X \mid x \notin f(x)\}$$

   Notice that $\Delta \subseteq X$, so that $\Delta \in \mathbb{P}(X)$ by definition. Since f is onto, this implies that there exists $\delta \in X$ such that $f(\delta) = \Delta$. We now have two cases.

   Case 1: Suppose that $\delta \in \Delta$. Then, we know that $\delta \notin f(\delta)$ by definition. However, since we know $f(\delta) = \Delta$, this means $\delta \notin \Delta$, contradicting our assumption in this case. ↯

   Case 2: Suppose that $\delta \notin \Delta$. Then, $\neg(\delta \notin f(\delta))$, implying $\delta \in f(\delta)$. Since $f(\delta) = \Delta$, this implies $\delta \in \Delta$, contradicting our assumption in this case. ↯

   Since we have a contradiction in either case (and these cases are exhaustive), we can therefore conclude that there is no onto function from X to $\mathbb{P}(X)$. Q.E.D.