Induction with Inequalities
Prove that $n^2 > 7n + 1$ for all integers $n \geq 8$

Solution:
Proof by induction on n
Base Case: $n = 8$

$$8^2 = 64 > 57 = 7 \cdot 8 + 1$$

Inductive Hypothesis: Assume that for all $8 \leq n < k, n^2 > 7n + 1$
Consider $n = k$
By the inductive hypothesis we get

$$\begin{aligned}(k-1)^2 &> 7(k-1) + 1 \\ k^2 - 2k + 1 &> 7k - 6 \\ k^2 &> 7k - 6 - (-2k + 1) \\ k^2 &> 7k + (2k - 7)\end{aligned}$$

Since $k \geq 8$

$$(2k - 7) \geq 2 \cdot 8 - 7 \geq 1$$

So

$$k^2 > 7k + (2k - 7) \geq 7k + 1$$

And thus by induction we have that $n^2 > 7n + 1$ for $n \geq 8$. $\square$

Big-O Analysis

Prove that 2^n is $O(n!)$

Solution:

To show that 2^n is $O(n!)$ we need to show that there are positive real numbers c and k such that $0 \leq 2^n \leq c \cdot n!$ for all $n > k$.

We will select $c = 1$ and $k = 4$ so we will prove the following.

So we now will prove $2^n \leq n!$ for all $n \geq 4$

Proof by induction on n

Base Case: $n = 4$

$$2^4 = 16 \leq 24 = 4!$$

Inductive Hypothesis: Assume that $2^n \leq n!$ for all $4 \leq n < j$

Consider $n = j$

$$j! = j(j-1)!$$

By Inductive hypothesis $(j-1)! \geq 2^{j-1}$ so

$$j! = j(j-1)! \geq j \cdot 2^{j-1}$$

Since $j \geq 4$ it holds that $j \geq 4 > 2$ so

$$j! = j(j-1)! \geq j \cdot 2^{j-1} > 2 \cdot 2^{j-1} > 2^j$$

Thus for all $2^n \leq n!$ for all $n \geq 4$ and thus 2^n is $O(n!)\square$

Big-O Analysis

Consider two functions $f(n)$ which is $O(2^n)$ and $g(n)$ which is $O(n!)$. Is it then the case that $f(n)$ is $O(g(n))$?

Solution:

This is not true. Consider $f(n) = 2^n$ which is clearly $O(2^n)$ and $g(n) = 1$ which is clearly $O(n!)$ but it is also clear that 2^n is not $O(1)$ so $f(n)$ is not necessarily $O(g(n))$.