Recursive Definition Suppose that $f : \mathbb{N} \to \mathbb{N}$ is defined by

$f(0) = 2$

$f(1) = 3$

$f(n) = 3f(n-1) - 2f(n-2)$ for all $n \geq 2$.

Solution: For this we first try several values to come up with a guess for the closed form of the equation.

$$f(2) = 3f(1) - 2f(0) = 3 \cdot 3 - 2 \cdot 2 = 5$$
$$f(3) = 3f(2) - 2f(1) = 3 \cdot 5 - 2 \cdot 3 = 9$$

So the guess I have here is that $f(n) = 2^n + 1$ to finish the problem we need to prove that this is true. We will do this by induction on n.

We need to start with bases cases as follows.

$f(0) = 2^0 + 1 = 2$ which matches the definition.

$f(1) = 2^1 = 3$ which matches the definition.

Assume that $f(n) = 2^n + 1$ for all $n < k$ as an inductive hypothesis.

Now consider $n = k$. Starting with the definition we have do the following.

$$f(k) = 3f(k-1) - 2f(k-2)$$

Since $k-1$ and $k-2$ are both less than k we can apply the inductive hypothesis and get the following.

$$\begin{aligned} f(k) &= 3(2^{k-1} + 1) - 2(2^{k-2} + 1) \\ &= 3 \cdot 2^{k-1} + 3 - 2 \cdot 2^{k-2} + 2 \\ &= 3 \cdot 2^{k-1} + 3 - 2^{k-1} + 2 \\ &= 2 \cdot 2^{k-1} + 1 \\ &= 2^k + 1 \end{aligned} \tag{1}$$

Thus with the base cases and inductive step we can conclude that $f(n) = 2^n + 1$. $\square$

Unrolling

$T : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by

$T(1) = 1$

$T(n) = 2T(n-1) + 3$

Solution:

$$\begin{aligned} T(n) &= 2T(n-1) + 3 \\ T(n) &= 2(2T(n-2) + 3) + 3 = 2^2T(n-2) + 2 \cdot 3 + 3 \\ T(n) &= 2(2(2T(n-3) + 3) + 3) + 3 = 2^3T(n-3) + 2^2 \cdot 3 + 2 \cdot 3 + 3 \end{aligned} \tag{2}$$

So we get the following as a general form for k.

$$T(n) = 2^kT(n-k) + \sum_{i=0}^{k-1} 2^i \cdot 3$$

To get a closed form we need to remove the k we do this by solving $n - k = 1$ to reach the base case of $T(1) = 1$ and thus get $k = n - 1$. Plugging this in we get the following.

$$T(n) = 2^{n-1}T(1) + \sum_{i=0}^{n-2} 2^i \cdot 3 = 2^{n-1} + 3\sum_{i=0}^{n-2} 2^i$$

Finally we look up a closed form for the summation and see that $\sum_{i=0}^{n-2} 2^i = 2^{n-1} - 1$ and plugging that in we get the following.

$$T(n) = 2^{n-1} + 3(3^{n-1} - 1) = 4(2^{n-1}) - 3 = 2^{n+1} - 3$$