Sets and Modular Arithmetic Tutorial Problems

1. Congruence classes of perfect squares

a) Compute $\{[x^2]_4 \mid x \in \mathbb{Z}\}$. *(That is, rewrite the set into a simpler form that lists all the elements explicitly.)*

b) Notice that, for any k, $[a]_k \neq [b]_k$ implies $a \neq b$. (Do you see why this is true?) Using this fact and the result from part (a), prove that for all integers x and y, $x^2 + y^2 \neq 4000003$. (Do not use a calculator.)

2. Sets warmup

Consider the following sets: $A = \{2\}$, $B = \{A, \{4, 5\}\}$, $C = B \cup \emptyset$, $D = B \cup \{\emptyset\}$.

a) Which of the sets have more than two elements?

b) Which of the following are true:

$$2 \in A,\ 2 \in B,\ \{2\} \in A,\ \{2\} \in B,\ \emptyset \in C,\ \emptyset \in D,$$
$$\emptyset \subseteq A,\ \{2\} \subseteq A,\ \{2\} \subseteq B$$

3. Cartesian product

a) Find an example of sets A and B such that $A \times B = B \times A$. Then find a second such pair of sets; try to make this second example feel *different* from your first, e.g. don't just rename some elements.

b) Consider the following incomplete statement:

For sets A and B, if ________________ then $A \times B \neq B \times A$.

Create a true claim by filling in the blank with a statement about A and B that does not mention Cartesian products. Try to make the *strongest* possible claim, i.e. ideally your statement should still be true even if we replaced the "if-then" by an "if and only if". *If you have extra time, also prove your claim. Hint: two sets are not-equal if and only if there exists an element that is in one but not the other.*