Induction-Like Implications[1]

Do not *use induction for this problem, but think about how this problem relates to the mechanics of induction.*

For each statement S below, answer the following two prompts:

- Find a predicate $P(n)$ on natural numbers where $(\forall k, P(k) \to P(k+2))$ is true and also S is true, or explain why no such predicate exists.
- Find a predicate $P(n)$ on natural numbers where $(\forall k, P(k) \to P(k+2))$ is true but S is false, or explain why no such predicate exists.

For example, if S is the statement "$\forall n, P(n)$", then:

- For the first prompt, we could define $P(n)$ to be "$n = n$" (or anything else that is true for every natural number[2]). This clearly makes S true, and it also makes the required induction-like implication true since for every k, $P(k) \to P(k+2) \equiv T \to T \equiv T$).
- For the second prompt, we could define $P(n)$ to be "n is even". This makes S false (as not all natural numbers are even), but our induction-like implication is still true: on even k it's $T \to T \equiv T$, and on odd k it's $F \to F \equiv T$.

c) $S =$ "$\forall n \geq 0, \neg P(n)$"

d) $S =$ "$(\forall n \leq 100, P(n)) \wedge (\forall n > 100, \neg P(n))$"

e) $S =$ "$(\forall n \leq 100, \neg P(n)) \wedge (\forall n > 100, P(n))$"

f) $S =$ "$P(0) \to \forall n, P(n+2)$"

h) $S =$ "$P(1) \to \forall n, P(2n+1)$"

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[1] This problem was adapted from "Mathematics for Computer Science" by Lehman et al. problem 5.16. `https://courses.grainger.illinois.edu/cs173/fa2020/Textbook/MITMathCS.pdf`

[2] Or more simply, you can just say $P(n)$ is defined to be "true", i.e. it ignores its argument and outputs true no matter what.