Lecture 8: Finite Cardinality and Induction

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Sequences on A: Ordered list of elements from A.

- Length two sequences (a_1, a_2), i.e., pairs, i.e., element of $A \times A$
- Length n sequences $(a_1, a_2, \dots a_n) \in A \times A \times \cdots A$

$A_1 \times A_2 \times \dots A_n = \{(a_1, a_2 \dots a_n) \mid \forall i .\ a_i \in A_i\}$

Bijective Functions:

- $f : A \to B$ is surjective/onto if $\text{range}(f) = f(A) = B = \text{codomain}(f)$.
- $f : A \to B$ is injective/1-to-1 if *distinct* elements get mapped to *distinct* elements.
- A function is bijective if it is injective/1-to-1 and surjective/onto.

Cardinality (of finite sets): $|X|$ = number of elements in X.

Example 1. $|\emptyset| = 0$ $\quad |\{0,1,2,3\}| = 4$ $\quad |\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}| = 4$ $\quad |\{0,1,1,2,2\}| = 3$

$|\{0,1,2\} \times \{a,b,c\}| =$ $\quad$ |sequences of length n over $\{0,1,2\}$| $= |\underbrace{\{0,1,2\} \times \{0,1,2\} \times \dots \{0,1,2\}}_{n}|$

$= |\{(0,a), (0,b), (0,c) \dots\}| = 9$ $\qquad = 3 \times 3 \times 3 \times \dots 3 = 3^n$

If A, B s.t $|A| = n$, $|B| = m$, $|A \times B| = mn$

Proposition 1. *The following statements hold for finite sets A and B.*

1. *If there is a surjective function $f : A \to B$ then $|A| \geq |B|$.*
2. *If there is a injective function $f : A \to B$ then $|A| \leq |B|$.*
3. *If there is a bijective function $f : A \to B$ then $|A| = |B|$.*

$f : A \to B$. $\quad |A| \geq |\underbrace{\{f(a) \mid a \in A\}}_{\subseteq B}| \leq |B|$

f is onto: $\quad |A| \geq |\underbrace{\{f(a) \mid a \in A\}}_{= B}| = |B|$

f is 1-to-1: $\quad |A| = |\{f(a) \mid a \in A\}| \leq |B|$

f is bijective: $|A| = |\{f(a) \mid a \in A\}| = |B|$

Proposition 2. *For a set A such that $|A| = n$, $|\text{pow}(A)| = 2^n$.*

$\text{pow}(A) = \{B \mid B \subseteq A\}$ $\qquad f : \text{pow}(A) \to \underbrace{\{0,1\} \times \{0,1\} \times \dots \{0,1\}}_{n}$

$A = \{a_1, a_2 \dots a_n\}$ $\qquad f(B) = (b_1, b_2 \dots b_n)$ s.t. $b_i = \begin{cases} 0 & \text{if } a_i \notin B \\ 1 & \text{if } a_i \in B \end{cases}$

$A = \{0, 1, 2 \dots n-1\}$.

$B = \{1, 4, 5\}$ $\quad f(B) = (0,1,0,0,1,1,0, \dots 0,)$ $\quad$ Claim: f is bijective

$\Rightarrow |\text{pow}(A)| = |n \text{ length binary seq}|$

Induction: To prove $\forall n \in \mathbb{N}\ P(n) \to P(0)$ AND $P(1)$ AND $P(2)$ . . . .

- Prove $P(0)$ **[Base Case]**
- Prove for all $n > 0$, if $P(0)$ AND $P(1)$ AND $\cdots$ AND $P(n-1)$ then $P(n)$ **[Induction Step]**

0 1 2 3 4 . . . .

$0+1+2 \cdots + n$

$n=0: \ 0$

$n=1: \ 0+1$

Proposition 3. *Prove for all $n \in \mathbb{N}$*

$$\overbrace{0+1+2+\cdots+n}^{\sum_{i=0}^{n} i} = \frac{n(n+1)}{2}$$

$\rightarrow P(n): \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ $\quad \forall n \in \mathbb{N} \ P(n)$

Base Case: $P(0)$: $0 = \frac{0(0+1)}{2}$ ✓

For any n. $P(0)$ AND $P(1)$ AND $\cdots$ $P(n-1)$ IMPLIES $P(n)$

Ind Hypothesis: $\sum_{i=0}^{k} i = \frac{k(k+1)}{2} \quad \forall k \leq n-1$

To prove $0+1+2+\cdots n-1+n = \frac{(n+1)n}{2}$ $\quad P(n-1): 0+1+\cdots n-1 = \frac{(n-1)(n)}{2}$

LHS: $0+1+2+\cdots n-1+n$

$= \frac{(n-1)n}{2} + n$ (ind hyp)

$= \frac{n(n-1)+2n}{2} = \frac{n(n-1+2)}{2} = \frac{n(n+1)}{2}$ = RHS.

Proposition 4. *Prove that for all $n \in \mathbb{N}$, $\sum_{i=0}^{n} i2^i = (n-1)2^{n+1} + 2$.*

Problem 1. All horses have the same color.

Proof by induction. Predicate $P(n)$: Any set of n-horses has the same color. To prove: $\forall n \in \mathbb{N}$ with $n \geq 1$, $P(n)$

Base Case: $P(1)$. In any set containing only one horse, all horses (namely the only one) have the same color.

Induction Hypothesis: Assume that $P(1), P(2), \ldots P(n-1)$ are true.

Induction Step: Consider an arbitrary set H of $n+1$ horses.

Let $H = \{h_1, h_2, \ldots h_n\}$

Consider $H_1 = \{h_1, h_2, \ldots h_{n-1}\}$ and $H_2 = \{h_2, \ldots h_n\}$

Since $P(n-1)$ holds, all horses in H_1 have the same color. Also all horses in H_2 have the same color.

So color(h_1) = color(h_2) = color(h_3) = $\cdots$ = color(h_n). Hence all horses in H have the same color.