warm-up: simplify $(p \to q) \wedge (\neg p \to q)$

$$\equiv (\neg p \vee q) \wedge (p \vee q)$$

$$\equiv (p \wedge \neg p) \vee q$$

$$\equiv \text{False} \vee q$$

$$\equiv q$$

Proof by cases:

$(p_1 \vee p_2 \vee \dots \vee p_k) \wedge (p_1 \to q) \wedge (p_2 \to q) \wedge \cdots (p_k \to q)$

implies

q

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warm-up: Simplify $\neg p \to \text{False}$

$$\equiv p \vee \text{False} \equiv p$$

$$\boxed{\neg p \to \text{False} \equiv p}$$

Case 4: Assume $y \geq 0 > x$. Then $|x| = -x$, $|y| = y$, $|xy| = -xy$. Hence, $|x||y| = -xy = |xy|$.

Lecture 6: More Proofs

September 9, 2019

Definition 1. For a real number x, $|x|$ is defined as follows.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$$

Problem 1. For real numbers x, y, $|xy| = |x||y|$.

We prove this by cases.

Case 1: Assume $x, y \geq 0$. Then: $|x| = x$, $|y| = y$ and $|xy| = xy$. Therefore, $|x||y| = xy = |xy|$.

Case 2: Assume $x, y < 0$. Then $|x| = -x$, $|y| = -y$, $|xy| = xy$. Therefore, $|x||y| = (-x)(-y) = xy = |xy|$.

Case 3: Assume without loss of generality that $x \geq 0 > y$. Then $|x| = x$, $|y| = -y$, $|xy| = -xy$. Therefore, $|x||y| = x(-y) = -xy = |xy|$.

Problem 2. Prove that $\sqrt{2}$ is irrational.

We prove this by contradiction. Assume that $\sqrt{2}$ is rational. Let $\sqrt{2} = \frac{a}{b}$ be a ratio in simplest form. $a, b \in \mathbb{Z}$, $a \geq 0$, and a and b have no prime factors in common. Then:

$\sqrt{2} = \frac{a}{b} \Rightarrow \sqrt{2}\, b = a \Rightarrow 2b^2 = a^2 \Rightarrow a^2$ is even

$\Rightarrow a$ is even $\Rightarrow a = 2k$ for some $k \in \mathbb{Z}$.

$\Rightarrow 2b^2 = (2k)^2 = 4k^2 \Rightarrow b^2 = 2k^2 \Rightarrow b^2$ is even

$\Rightarrow b$ is even

~~This~~ a being even and b being even contradicts our assumption that a/b is in simplest form. Hence, $\sqrt{2}$ is irrationa

Problem 3. There are infinitely many primes.

We prove this by contradiction. Assume that there only finitely many primes. Let k be the number of primes and let $p_1, \dots, p_k$ the k prime numbers. Consider $m := (\prod_{i=1}^{k} p_i) + 1$. m is not divisible by p_i for $i \in \{1, \dots, k\}$. Therefore, m is a prime number greater than $p_1, p_2, \dots, p_k$. This contradicts our assumption that there are only k primes. Therefore, there must infinitely many primes.

Problem 4. There are irrational numebrs x and y such that x^y is rational.

$w = \sqrt{2}^{\sqrt{2}}$

case 1: w is rational. Then $x = y := \sqrt{2}$ satisfies the requirements.

case 2: w is irrational. Consider $w^{\sqrt{2}}$

$$w^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \times \sqrt{2}} = \sqrt{2}^{2} = 2$$

In this case $x := w$, $y := \sqrt{2}$ satisfies the requirements

$$m = \left(\prod_{i=1}^{k} p_i\right) + 1$$

$$m = p_1 \left(\prod_{i=2}^{k} p_i\right) + 1$$

$$m = p_1 \times s + 1$$

remainder of m to p_1 is 1.