Lecture 35: Infinite Sets

Date: December 4, 2019.

Comparing the sizes of Infinite sets.

Proposition 1. *The following statements hold for finite sets A and B.*

1. *If there is a surjective function $f : A \to B$ then $|A| \geq |B|$.*
2. *If there is a injective function $f : A \to B$ then $|A| \leq |B|$.*
3. *If there is a bijective function $f : A \to B$ then $|A| = |B|$.*

Cantor's Definition. For infinite sets A and B, we will say $|A| \leq |B|$ if there is a injective function $f : A \to B$.

- We will say $|A| = |B|$ if there is a bijective function $f : A \to B$.

Proposition 2. *For any (non-empty) sets A, B, and C the following properties hold.*

1. *If there are injective functions $f : A \to B$, and $g : B \to C$ then there is an injective function $h : A \to C$.*
2. *If there is an injective function $f : A \to B$ then there is a surjective function $g : B \to A$.*

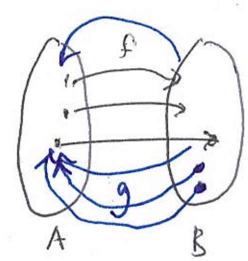
1. $h = g \circ f$. $\quad g \circ f : A \to C \quad g \circ f(a) = g(f(a))$

If f and g are injective then $g \circ f$ is also injective.

$(|A| \leq |B|$ and $|B| \leq |C|)$ IMPLIES $|A| \leq |C|$.

2. Let a_0 be an arbitrary element of A.

$$g(b) = \begin{cases} a & \text{if } f(a) = b \\ a_0 & \text{if } b \notin \operatorname{img}(f). \end{cases}$$



Theorem 3 (Cantor-Schröder-Bernstein). *For any sets A and B, if there are injective functions $f : A \to B$ and $g : B \to A$ then there is a bijective function $h : A \to B$.*

If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

Problem 1. Let $\mathbb{E} = \{n \in \mathbb{N} \mid n \equiv 0 \pmod 2\}$. Show that $|\mathbb{E}| \le |\mathbb{N}|$ and $|\mathbb{N}| \le |\mathbb{E}|$.

$\mathbb{E} \subsetneq \mathbb{N}$ $\quad id : \mathbb{E} \to \mathbb{N}. \quad id(a) = a \ \forall a \in \mathbb{E}$

$dbl : \mathbb{N} \to \mathbb{E}$

$dbl(x) = 2x. \ \forall x \in \mathbb{N}.$ injective.

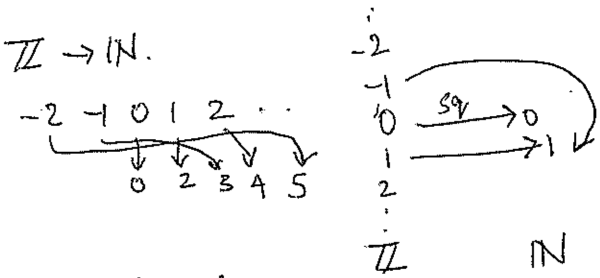
$|\mathbb{E}| = |\mathbb{N}|$

If A and B are finite sets and $A \subsetneq B$ then $|A| \neq |B|$

Problem 2. Show that $|\mathbb{Z}| \le |\mathbb{N}|$.

Need to define injective fn $f : \mathbb{Z} \to \mathbb{N}$.

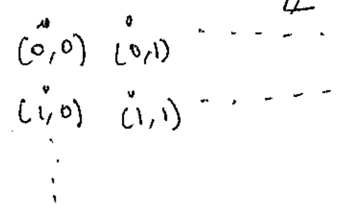
$$f(x) = \begin{cases} 2x & x \geq 0 \\ \underbrace{-2x+1}_{-2x-1} & x < 0 \end{cases}$$



Problem 3. Show that $|\mathbb{N} \times \mathbb{N}| \le |\mathbb{N}|$.

$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}.$

$f(a,b) = 2^a 3^b.$



Problem 4. Show that $|\mathbb{Q}| \le |\mathbb{N}|$.

Countable Sets. A (finite or infinite) set A is said to be **countable** if there is an injective function $f : A \to \mathbb{N}$. In other words, if $|A| \le |\mathbb{N}|$.

Theorem 4 (Cantor). *For any set A,* $\mathsf{pow}(A) \not\le A$

$|\mathbb{N}| < |pow(\mathbb{N})| < |pow(pow(\mathbb{N}))| \ \ldots$

Need to show: there is no injective function $f : pow(A) \to A$.

Instead prove: There is no surjective function $f : A \to pow(A)$.

Consider any $f : A \to pow(A)$. To show that f is not surjective

$S_f = \{a \in A \mid a \notin f(a)\}$

$0 \xrightarrow{f} \{1,2,3\}$

$1 \xrightarrow{f} \emptyset$

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Claim that $S_f \notin img(f)$

$\forall a.\ f(a) \neq S_f$. i.e $(f(a) - S_f) \cup (S_f - f(a)) \neq \emptyset$

$\left.\begin{array}{l} \underline{a \in f(a)} : \Rightarrow a \notin S_f \\ \underline{a \notin f(a)} : \Rightarrow a \in S_f. \end{array}\right\}$ $\quad a \in (f(a) - S_f) \cup (S_f - f(a)) \neq \emptyset$

$f(a) \neq S_f.$