Lecture 26: Sums, Products, and Bijections

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Sum Rule. If $A_1, A_2, \dots A_n$ are pairwise disjoint sets (i.e., $A_i \cap A_j = \emptyset$ for every $i \neq j$) then

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|.$$

Problem 1. Suppose we roll a black die and a white die. In how many outcomes will the two dice show different values?

A_i - set outcomes when black die shows i and the white shows a value $\neq i$

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| + |A_6|$$
$$= 5+5+5+5+5+5 = 30$$

Complementary Counting. Suppose $A \subseteq U$ To find $|A|$, sometimes it is easier to find $|U|$ and $|U - A|$; then $|A| = |U| - |U - A|$. $\Rightarrow$ A and $U-A$ are disjoint and $A \cup (U-A) = U$

U - set of all outcomes $= \{1,2 \dots 6\} \times \{1,2, \dots 6\}$. $|U| = 6 \times 6 = 36$

B - set of outcomes where black = white $|B| = 6$

A - set of outcomes when black $\neq$ white. $|A| = |U| - |B| = 36 - 6 = 30$

Product Rule. If $A_1, A_2, \dots A_n$ are finite sets, then

$$|A_1 \times A_2 \times \cdots \times A_n| = \prod_{i=1}^{n} |A_i|.$$

Problem 2. How many binary strings of length n?

$|\{0,1\} \times \{0,1\} \times \cdots \{0,1\}| = 2 \times 2 \times 2 \times \cdots 2 = 2^n$.

Problem 3. A restaurant menu has 5 appetizers, 6 entrees, 3 salads, and 7 desserts.

1. How many items are on the menu?
   $5+6+3+7 = 21$

   A - appetizers S - salads
   E - entrees D - desserts.

2. How many ways to choose a complete meal that consists of each course?
   $|A \times E \times S \times D| = 5 \times 6 \times 3 \times 7 = 630$

3. How many ways to order a meal if I may not choose some courses?
   $|(A \cup \{\text{nothing}\}) \times (E \cup \{\text{nothing}\}) \times (S \cup \{\text{nothing}\}) \times (D \cup \{\text{nothing}\})| = 6 \times 7 \times 4 \times 8 = 1344$

Problem 4. Suppose we roll a black die and a white die. In how many outcomes will the black die show a smaller value than the white die?

A_i - outcomes where black $= i$ and white $>$ black

$$|\bigcup_{i=1}^{6} A_i| = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$$
$$= 5+4+3+2+1+0 = 15$$

G - set outcomes where Black $<$ White

L - set outcomes where Black $>$ White

① $G \cup L$ - outcomes Black $\neq$ White

$|G \cup L| = 30 = |G| + |L|$

② $|G| = |L| \Rightarrow |G| = 15$.

$f: G \to L$ $G = \{(b,w) \mid b < w\}$

$L = \{(b,w) \mid b > w\}$

$f((b,w)) = (w,b)$

f is bijective.

Correspondence Principle. For finite sets A and B

- If there is a surjection $f : A \to B$ then $|A| \geq |B|$.
- If there is a injection $f : A \to B$ then $|A| \leq |B|$.
- If there is a bijection $f : A \to B$ then $|A| = |B|$.

Proposition 1. *Number of subsets of a set A of size n is 2^n.*

Need compute $|pow(A)|$. $\exists$ bijection $f : pow(A) \to$ Binary strings of length n.
$A = \{a_1, a_2 \ldots a_n\}$. $f(S) = b_1 b_2 \ldots b_n$ s.t. $b_i = 1$ iff $a_i \in S$.

Problem 5. A valid password is a sequence between 6 and 8 symbols. The first symbol must be a letter (upper or lower case) and the remaining symbols can either be a letter (upper or lower case) or a digit. How many passwords are there?

L = set of letters $|L| = 52$ S – set of letter + digits $|S| = 62$.

$P = L \times S^5 \cup L \times S^6 \cup L \times S^7$

$|P| = |L \times S^5| + |L \times S^6| + |L \times S^7| = 52 \times 62^5 [1 + 62 + 62^2]$

$S' = S \cup \{\text{nothing}\}$

$P = L \times S^5 \times S' \times S'$.

Generalized Product Rule. Let S be a set of length k sequences such that there are n_1 possibilities for the first entries, n_2 possibilities for the second entries for each first entry, $\ldots n_k$ possibilities for the kth entries for each sequence of first $k-1$ entries. Then $|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$.

Problem 6. How many ways to order a deck with 52 cards?

$52 \times 51 \times 50 \times \ldots \times 1 = 52!$

Problem 7. A dollar bill is *defective* if some digit appears more than once in the 8-digit serial number. How many defective bills are there?

How many ~~regular~~ serial numbers – $10 \times 10 \times 10 \cdots \times 10 = 10^8$.

Good serial numbers $= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3$

Defective serial numbers $= 10^8 - (10 \times 9 \times 8 \times \cdots \times 3)$