Lecture 25: Recurrences

Date: November 1, 2019.

Recurrences: An infinite sequence of numbers $a_0, a_1, \ldots$ where the nth element of the sequence is defined in terms of other elements in the sequence, and the first few terms are given explictly. The goal is to find "closed form" for the nth element in the sequence.

$f_0 = 0$, $f_1 = 1$. $\quad f_n = f_{n-1} + f_{n-2} \quad n \geq 2$

Problem 1. Show that $T_n = 2^n - 1$ satisfies the recurrence: $T_0 = 0$ and $T_n = 2T_{n-1} + 1$ when $n \geq 1$.

Guess & verify: Induction. $P(n)$: $T_n = 2^n - 1$

Plug & Chug:

$$\left.\begin{aligned} T_n &= 2T_{n-1} + 1 \\ &= 2(2T_{n-2}+1)+1 = 2^2 T_{n-2} + 2 + 1 \\ &= 2^2(2T_{n-3}+1) + 2 + 1 = 2^3 T_{n-3} + 2^2 + 2 + 1 \\ &= 2^3(2T_{n-4}+1) + 2^2 + 2 + 1 = 2^4 T_{n-4} + 2^3 + 2^2 + 2 + 1 \end{aligned}\right\} \text{Unrolling Plugging}$$

Guess: $T_n = 2^k T_{n-k} + \sum_{i=0}^{k-1} 2^i = 2^k T_{n-k} + \frac{(2^k - 1)}{2-1} = 2^k T_{n-k} + 2^k - 1$

Verify: Suppose $T_n = 2^k T_{n-k} + 2^k - 1$

$$T_n = 2^k[2T_{n-(k+1)} + 1] + 2^k - 1 = 2^{k+1} T_{n-(k+1)} + 2^k + 2^k - 1$$
$$= 2^{k+1} T_{n-(k+1)} + 2^{k+1} - 1$$

Step 3 $T_n = 2^n T_0 + 2^n - 1 = 2^n - 1$

Problem 2. Solve the recurrence given by: $T_1 = 0$ and $T_n = 2T_{n/2} + n - 1$ when $n \geq 2$ and a power of 2.

$$\begin{aligned} T_n &= 2T_{n/2} + n - 1 \\ &= 2\left[2T_{n/4} + \frac{n}{2} - 1\right] + n - 1 = 4T_{n/4} + n - 2 + n - 1 \\ &= 4\left[2T_{n/8} + \frac{n}{4} - 1\right] + n + n - 2 - 1 = 8T_{n/8} + n + n + n - 4 - 2 - 1 \\ &= 2^3 T_{n/2^3} + 3n - [2^2 + 2^1 + 2^0] \end{aligned}$$

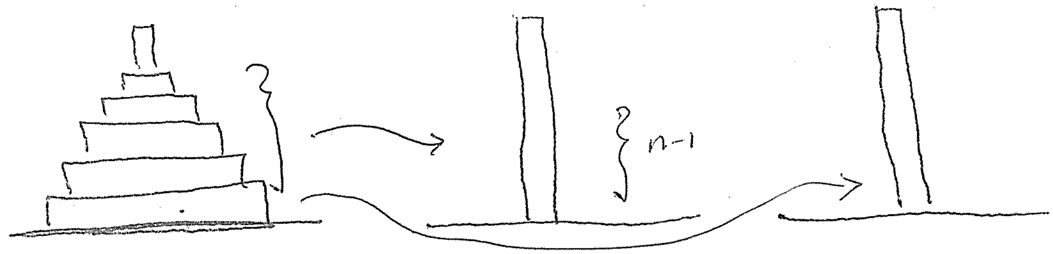
Guess: $T_n = 2^k T_{n/2^k} + kn - [2^k - 1]$

Verify:
$$\begin{aligned} T_n &= 2^k T_{n/2^k} + kn - [2^k - 1] \\ &= 2^k\left[2T_{n/2^{k+1}} + \left[\frac{n}{2^k}\right] - 1\right] + kn - [2^k - 1] \\ &= 2^{k+1} T_{n/2^{k+1}} + n - 2^k + kn - [2^k - 1] \\ &= 2^{k+1} T_{n/2^{k+1}} + (k+1)n - [2^{k+1} - 1] \end{aligned}$$

Taking $k = \log_2 n$

$$T_n = 2^{\log_2 n} T_1 + (\log_2 n)\, n - [2^{\log_2 n} - 1] = nT_1 + n\log n - [n-1]$$
$$= n\log n - n + 1$$

TOWERS OF HANOI



Move disks from one peg to another s.t.

- Disks moved one at a time
- Larger disk never placed on top of smaller disk

Problem 3. Find a closed form solution to: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

$f_n = x^n$ is a possible form of the solution.
Substituting x^n for f_n we get.

$x^n = x^{n-1} + x^{n-2}$

Assume $x \neq 0$, $\quad x^2 = x + 1$

Root of $x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{1+4}}{2}$

$x = \frac{1+\sqrt{5}}{2} \qquad x = \frac{1-\sqrt{5}}{2}$

Proposition: Suppose $g(n)$ and $h(n)$ satisfy $g(n) = g(n-1) + g(n-2)$ and $h(n) = h(n-1) + h(n-2)$

Then for any $s, t \in \mathbb{R}$. "$sg(n) + th(n)$" also satisfies the recurrence

$$sg(n) + th(n) = s[g(n-1) + g(n-2)] + t[h(n-1) + h(n-2)]$$
$$= (s\,g(n-1) + t\,h(n-1)) + (sg(n-2) + t\,h(n-2))$$

In general, $s\left(\frac{1+\sqrt{5}}{2}\right)^n + t\left(\frac{1-\sqrt{5}}{2}\right)^n$ is a solution to $f_n = f_{n-1} + f_{n-2}$

$f_0 = 0 = s\left(\frac{1+\sqrt{5}}{2}\right)^0 + t\left(\frac{1-\sqrt{5}}{2}\right)^0 = s + t.$

$f_1 = 1 = s\left(\frac{1+\sqrt{5}}{2}\right) + t\left(\frac{1-\sqrt{5}}{2}\right) = s\left[\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)\right] = 1$

$s = \frac{1}{\sqrt{5}} \Rightarrow t = \frac{-1}{\sqrt{5}}.$

$$f_n = \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{\phi\text{ - golden ratio}}^n - \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{\to \frac{1}{\phi}}^n$$

Homogeneous Linear Recurrence: is of the form

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \cdots + a_d f(n-d).$$

Substituting $F(n) = x^n$ (with $x \neq 0$) and simplifying, we get the **characteristic equation**

$$x^d = a_1 x^{d-1} + a_2 x^{d-2} + \cdots a_d.$$

Roots of the characteristic equation define solutions

- If r is a non-repeated root then r^n is a solution to the recurrence.
- If r is a repeated root with multiplicity k then $r^n, nr^n, n^2r^n, \ldots n^k r^n$ are all solutions.

General solution is a linear combination of all solutions identified above. Use boundary conditions to find coefficients of the general solution, to get a particular solution that satisfies the linear recurrence and the boundary conditions.