Lecture 24: Summing Series

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Arithmetic Progression. A sequence of (real) numbers such that difference between successive elements is the same. It is of the form

$$a, a+d, a+2d, \dots a+nd, \dots$$

a is the **initial term** and d is the **common difference.**

Proposition 1. *Let $a_1, a_2, \dots a_n$ be an arithmetic progression with initial term a and common difference d. Then $a_i = a + (i-1)d$, and*

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (a + (i-1)d) = \frac{n(a_1 + a_n)}{2} = \frac{n(2a + (n-1)d)}{2}.$$

$$S = a_1 + (a_1 + d) + \dots + (a_n - d) + a_n$$
$$S = a_n + (a_n - d) + \dots + (a_1 + d) + a_1$$
$$2S = (a_1 + a_n) + (a_1 + a_n) + \dots + (a_1 + a_n) + (a_1 + a_n) = n(a_1 + a_n)$$
$$S = \frac{n}{2}(a_1 + a_n)$$
$$= \frac{n}{2}(a + a + (n-1)d) = \frac{n}{2}(2a + (n-1)d) \qquad (a_1 = a,\ a_n = a + (n-1)d)$$

Corollary 2. $\sum_{i=1}^{n} i = n(n+1)/2$

Geometric Progression. A sequence of (real) numbers such the ratio of successive elements is the same. It is of the form

$$a, ar, ar^2, \dots ar^n, \dots$$

a is the **initial term** and r is the **common ratio.**

Proposition 3. *Let $a_1, a_2, \dots a_n$ be a geometric progression with initial term a and common ratio r. Then $a_i = ar^{i-1}$. If $r = 1$, $\sum_{i=1}^{n} a_i = na$. If $r \neq 1$,*

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n - 1)}{r - 1}.$$

<u>Case $r = 1$</u>: $a + ar + \dots ar^{n-1} = \underbrace{a + a + a + \dots a}_{n \text{ terms}} = na.$

<u>Case $r \neq 1$</u>:

$$S = a + ar + \dots ar^{n-1}$$
$$rS = ar + ar^2 \dots ar^{n-1} + ar^n$$
$$(1-r)S = a - ar^n = a(1 - r^n)$$
$$S = \frac{a(1-r^n)}{1-r} = \frac{a(r^n - 1)}{r - 1}$$

Proposition 4. *If* $|r| < 1$ *then*

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$$

$$\sum_{i=0}^{\infty} a r^i = \lim_{n\to\infty}\left(\sum_{i=0}^{n} a r^i\right)$$

$$= \lim_{n\to\infty}\left[\frac{a\left[r^{n+1}-1\right]}{r-1}\right] = \lim_{n\to\infty} \frac{a\left(1-r^{n+1}\right)}{1-r}$$

$$= \frac{a}{1-r}$$

Proposition 5.

$$\sum_{i=1}^{n} ix^i = \frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}$$

$$S = x + 2x^2 + 3x^3 + \cdots + nx^n$$

$$xS = x^2 + 2x^3 + \qquad + (n-1)x^n + nx^{n+1}$$

$$(1-x)S = x + x^2 + x^3 + \qquad + x^n - nx^{n+1}$$

$$= \frac{x(1-x^n)}{1-x} - nx^{n+1}$$

$$S = \frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}.$$

If $|x| < 1$ $\quad \sum_{i=0}^{\infty} ix^i = \frac{x}{(1-x)^2}$

$$\sum_{i=0}^{\infty} ix^i = \lim_{n\to\infty}\left(\sum_{i=0}^{n} ix^i\right) = \lim_{n\to\infty}\left[\frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}\right]$$

$$= \frac{x}{(1-x)^2} \quad \text{because } \boxed{\lim_{n\to\infty} n x^{n+1} = 0}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n}(i+1)^3 = \sum_{i=0}^{n}\left(i^3 + 3i^2 + 3i + 1\right)$$

$$\underbrace{\sum_{i=0}^{n}(i+1)^3}_{1^3+2^3+\cdots(n+1)^3} = \underbrace{\sum_{i=0}^{n} i^3}_{0^3+1^3+\cdots n^3} + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + (n+1)$$

$$3\sum_{i=0}^{n} i^2 = (n+1)^3 - 3\sum_{i=0}^{n} i - (n+1)$$