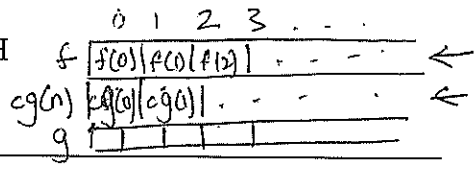


LECTURE 23: BIG OH

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Big Oh. For $f, g : \mathbb{N} \rightarrow \mathbb{N}$, we say that $f = O(g)$ (f is asymptotically at most as large as g) iff there is c, k such that for every $n \geq k$, $f(n) \leq cg(n)$.

Problem 1. Show that $\frac{(n-1)n}{2} = O(n^2)$ and $n^2 = O(\frac{(n-1)n}{2})$.

To prove: $\frac{(n-1)n}{2} = O(n^2)$ Need c, k s.t. $\forall n \geq k, \frac{n(n-1)}{2} \leq cn^2$
 $k=0, c=1$. For an arbitrary $n, \frac{n(n-1)}{2} \leq n^2 \quad | \quad k=10^6, c=10^6$.

To prove: $n^2 = O(\frac{n(n-1)}{2})$. $k=4, c=4$. When $n \geq 4, n^2 - 2n \geq 0$

Proposition 1. For any k , and $a_0, a_1, \dots, a_k, \sum_{i=0}^k a_i x^i = O(x^k)$.

$a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k$

$k=0, c = |a_0| + |a_1| + \dots + |a_k| = \sum_{i=0}^k |a_i|$ if $x \geq 0$. Then $x^i \leq x^k$ for any $i \leq k$.

$a_0 + a_1 x + \dots + a_k x^k \leq |a_0| x^k + |a_1| x^k + \dots + |a_k| x^k = (\sum_{i=0}^k |a_i|) x^k$.

Proposition 2. If $f = O(h)$ and $g = O(h)$ then $f \pm g = O(h)$.

Assume $f = O(h), g = O(h) \Rightarrow \exists k_1, k_2, c_1, c_2$ s.t.
 $\forall n \geq k_1, f(n) \leq c_1 h(n)$ and $\forall n \geq k_2, g(n) \leq c_2 h(n)$.

Take $k = \max(k_1, k_2), c = |c_1| + |c_2|$

$\forall n \geq k, f(n) \pm g(n) \leq c_1 h(n) \pm c_2 h(n) \leq (|c_1| + |c_2|) h(n) = c h(n)$

Common Tips and Pitfalls

- Ignore constant factor and constant additions
- If each term in some expression (sum of things) is $O(h)$ then whole thing is $O(h)$.

When $f = O(g) \leftarrow g$ is an upper bound of f .
 $n^2 = O(2^n)$ Note: $2^n \neq O(n^2)$

- ① $n = O(n) = 2n \rightarrow$ Never write $O(g) = f$.
- ② Any constant function $17 = O(1)$
- ③ $\sum_{i=0}^n i = \sum_{i=1}^n O(1) = O(n) \times$ In fact $\sum_{i=0}^n i = \frac{n(n+1)}{2} = O(n^2)$
- ④ $4^n = O(2^n) \times$
- If $f = O(g)$ then $2^f = O(2^g) \times$

$T(n) = n^2 + 2^n + O(\frac{1}{n}) \rightarrow T$ is a function s.t. for some $f = O(\frac{1}{n})$
 $T = n^2 + 2^n + f(n)$.

Theta Notation. For function $f, g : \mathbb{N} \rightarrow \mathbb{N}$, $f = \Theta(g)$ iff $f = O(g)$ and $g = O(f)$.

$$\frac{n(n-1)}{2} = \Theta(n^2)$$

Little Oh. For functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$, we say $f = o(g)$ (f is asymptotically smaller than g) iff

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$$

$$n^2 = o(2^n)$$

But $n^2 \neq \Theta(2^n)$

Problem 2. Show that $n^2 = o(2^n)$.

$$\lim_{n \rightarrow \infty} \frac{n^2}{2^n} = \lim_{n \rightarrow \infty} \frac{n^2}{(1+1)^n} = \lim_{n \rightarrow \infty} \frac{n^2}{1^n + \binom{n}{1}1^{n-1} + \dots + \binom{n}{n}} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1+n}{n^2} + \frac{n(n-1)}{2} + \frac{n^3}{n^2} \dots}$$

$$= 0$$

Big Omega. For functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$, we say $f = \Omega(g)$ (f is asymptotically at least as large as g) iff there are c, k such that for all $n \geq k$, $f(n) \geq cg(n)$.

Proposition 3. $f = \Omega(g)$ iff $g = O(f)$.