Lecture 23: Big Oh

$$\begin{array}{c|c|c|c|c|c}
 & 0 & 1 & 2 & 3 & \dots \\
f & f(0) & f(1) & f(2) & \dots & \leftarrow \\
cg(n) & cg(0) & cg(1) & \dots & & \leftarrow \\
g & & & & &
\end{array}$$

Date: October 25, 2019.

Big Oh. For $f, g : \mathbb{N} \to \mathbb{N}$, we say that $f = O(g)$ (f is **asymptotically at most as large** as g) iff there is c, k such that for every $n \geq k$, $f(n) \leq cg(n)$.

Problem 1. Show that $\frac{(n-1)n}{2} = O(n^2)$ and $n^2 = O(\frac{(n-1)n}{2})$.

To prove: $\frac{(n-1)n}{2} = O(n^2)$ Need c, k s.t $\forall n \geq k$ $\frac{n(n-1)}{2} \leq cn^2$

$k = 0$, $c = 1$. For an arbitrary n, $\frac{n(n-1)}{2} \leq n^2$ | $k = 10^6$, $c = 10^6$.

To prove: $n^2 = O(\frac{n(n-1)}{2})$. $k = 4$, $c = 4$. When $n \geq 4$, $n^2 - 2n \geq 0$

$n \geq 4$. $n^2 \leq n^2 + (n^2 - 2n) = 4\left[\frac{n(n-1)}{2}\right]$

Proposition 1. *For any k, and $a_0, a_1, \dots a_k$, $\underbrace{\sum_{i=0}^k a_i x^i}_{a_0 + a_1 x + a_2 x^2 + \dots a_k x^k} = O(x^k)$.*

$k = 0$, $c = |a_0| + |a_1| \dots |a_k| = \sum_{i=0}^{k} |a_i|$ if $x \geq 0$. Then $x^i \leq x^j$ for any $i \leq j$.

$a_0 + a_1 x + \dots a_k x_k \leq |a_0| x^k + |a_1| x^k + \dots |a_k| x^k = (\sum_{i=0}^k |a_i|) x^k$.

Proposition 2. *If $f = O(h)$ and $g = O(h)$ then $f \pm g = O(h)$.*

Assume $f = O(h)$, $g = O(h) \Rightarrow \exists k_1, k_2, c_1, c_2$ s.t.

$\forall n \geq k_1$. $f(n) \leq c_1 h(n)$ and $\forall n \geq k_2$ $g(n) \leq c_2 h(n)$.

Take $k = \max(k_1, k_2)$, $c = |c_1| + |c_2|$

$\forall n \geq k$, $f(n) \pm g(n) \leq c_1 h(n) \pm c_2 h(n) \leq |c_1| h(n) + |c_2| h(n) = (|c_1| + |c_2|) h(n)$

Common Tips and Pitfalls

- Ignore constant factor and constant additions
- If each term in some expression (sum of things) is $O(h)$ then whole thing is $O(h)$.

When $f = O(g) \leftarrow g$ is an upper bound of f.

$n^2 = O(2^n)$ Note: $2^n \neq O(n^2)$

① $n = O(n) = 2n \longrightarrow$ Never write $O(g) = f$.

② Any constant function $17 = O(1)$

③ $\sum_{i=0}^{n} i = \sum_{i=1}^{n} O(1) = O(n)$ ✗ In fact $\sum_{i=0}^{n} i = \frac{n(n+1)}{2} = O(n^2)$

④ $4^n = O(2^n)$ ✗

If $f = O(g)$ then $2^f = O(2^g)$ ✗

$T(n) = n^2 + 2^n + O(\frac{1}{n})$ $\longrightarrow$ T is a function s.t $T = n^2 + 2^n + f(n)$ for some $f = O(\frac{1}{n})$.

Theta Notation. For function $f, g : \mathbb{N} \to \mathbb{N}$, $f = \Theta(g)$ iff $f = O(g)$ and $g = O(f)$. $\frac{n(n-1)}{2} = \Theta(n^2)$

Little Oh. For functions $f, g : \mathbb{N} \to \mathbb{N}$, we say $f = o(g)$ (f is **asymptotically smaller** than g) iff

$n^2 = O(2^n)$

But $n^2 \neq \Theta(2^n)$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

Problem 2. Show that $n^2 = o(2^n)$.

$$\lim_{n\to\infty} \frac{n^2}{2^n} = \lim_{n\to\infty} \frac{n^2}{(1+1)^n} = \lim_{n\to\infty} \frac{n^2}{1^n + \binom{n}{1}1^{n-1} + \cdots \binom{n}{n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{n^2} + \frac{n}{n^2} + \frac{\frac{n(n-1)}{2}}{n^2} + \frac{n^3}{n^2} \cdots}$$

$$= 0$$

Big Omega. For functions $f, g : \mathbb{N} \to \mathbb{N}$, we say $f = \Omega(g)$ (f is **asymptotically at least as large** as g) iff there are c, k such that for all $n \geq k$, $f(n) \geq cg(n)$.

Proposition 3. $f = \Omega(g)$ *iff* $g = O(f)$.