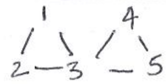
Lecture 20: Bipartite Graphs, and Coloring

Date: October 18, 2019.

Special Walks and Tours



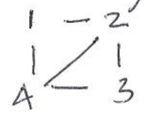
1{1,2} 2{2,3}3{3,4}4{4,5}5{5,3}3{3,1}1

Eulerian Tour of G is a closed walk that includes every edge exactly once.

Theorem 1. *A connected graph has an Eulerian tour if and only if every vertex has an even degree.*

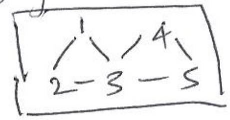
(⇒) Assume G has an Eulerian tour $v_0\{v_0,v_1\}v_1\{v_1,v_2\}v_2\{v_2,v_3\}v_3 \dots v_n$

Every time a vertex is visited we add 2 to degree. Degree is even.



1{1,2}2{2,4}4{4,1}1 ← Not Hamiltonian

1{1,2}2{2,3}3{3,4}4{4,1}1 Hamiltonian cycle

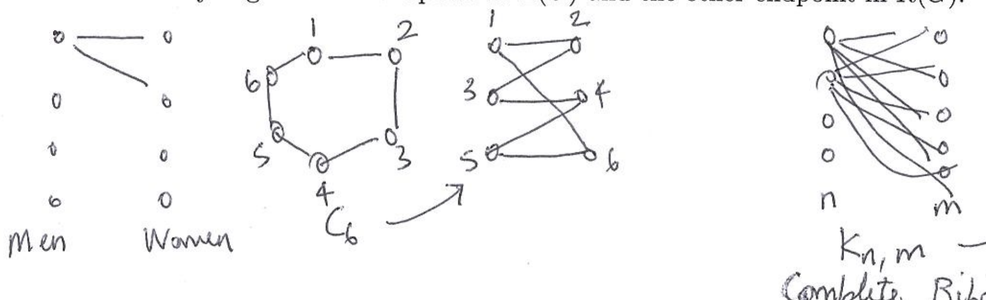


Hamiltonian Cycle of G is a cycle that visits every vertex in G exactly once.

Bipartite Graphs

$L(G) \cap R(G) = \emptyset$ AND $L(G) \cup R(G) = V(G)$

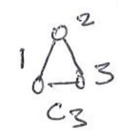
Definition. A graph G is **bipartite** if the set of vertices $V(G)$ can be *partitioned* into sets $L(G)$ and $R(G)$ such that every edge has one endpoint in $L(G)$ and the other endpoint in $R(G)$.



$V(K_{n,m}) = \{(0,1),(0,2) \dots (0,n)$
$(1,1),(1,2) \dots (1,m)\}$

$E(K_{n,m}) = \{\{(a,b),(c,d)\} \mid a \neq c\}$

Proposition 2. *Every cycle in a bipartite graph has even length.*



Let G to be bipartite w.r.t bipartition $L(G)$, $R(G)$.

Consider some cycle $\underbrace{v_0\{v_0,v_1\}v_1\{v_1,v_2\} \dots v_{n-1}}_{\text{distinct}}\{v_{n-1},v_n\}v_n$

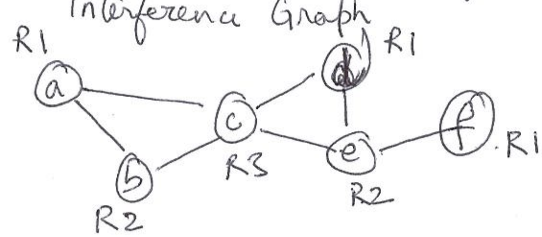
let $v_0 \in L(G) \Rightarrow v_1 \in R(G) \Rightarrow v_2 \in L(G) \dots$ | n is even. n edges.

Register Allocation

```
1 read a
2 read b
3 if (a > b)
4     c = a-b
5 else c = b-a
6 d = c
7 e = d + c
8 f = e/2
```

a : 1 — 5
b : 2 — 5
c : 4 — 7
d : 6 — 7
e : 7 — 8
f : 8.

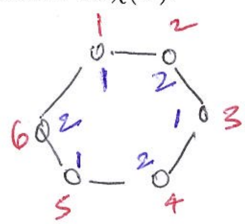
Interference Graph



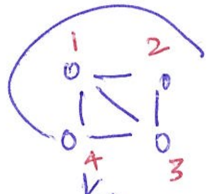
Coloring

A k-**coloring** of a graph G is $c : V(G) \to \{1, 2, \ldots k\}$ such that for any edge $\{u, v\} \in E(G)$, $c(u) \neq c(v)$.

Chromatic number. The least k such that G has a k-coloring is the **chromatic number** of G. It is denoted as $\chi(G)$.

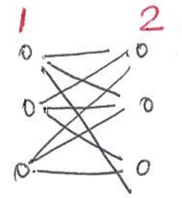


$\chi(G) = 2$



K_4

$\chi(K_n) = n$



$K_{3,3}$

Theorem 3. *A graph G with at least one edge is bipartite if and only if $\chi(G) = 2$.*

G is bipartite then it has a 2-coloring.

$\Rightarrow \chi(G) = 2$

Suppose G has a 2-coloring c.

Take $L(G) = \{u \mid c(u) = 1\}$ $\quad R(G) = \{u \mid c(u) = 2\}$

For any edge $\{u, v\} \in E(G)$, $c(u) \neq c(v)$

$\Rightarrow \{u, v\}$ has one endpoint in $L(G)$ and the other $R(G)$.

Theorem 4. *Let G be a graph such that for every vertex u, $\mathsf{deg}(u) \leq n$. Then $\chi(G) \leq n + 1$.*