Lecture 19: Subgraphs and Connectivity

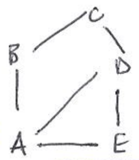
Date: October 16, 2019.

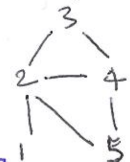
Isomorphism

Definition. An **isomorphism** between graphs G and H is a bijection $f : V(G) \to V(H)$ such that

$$\{u, v\} \in E(G) \text{ IFF } \{f(u), f(v)\} \in E(H).$$

G and H are said to **isomorphic** if there is (some) isomorphism between G and H.





$N_G(u) = \{v \mid \{u,v\} \in E(G)\}$ $\deg(u) = |N_G(u)|$

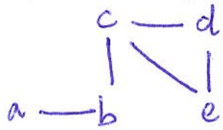
$N_H(f(u)) = \{f(v) \mid \{u,v\} \in E(G)\}$ ← If f is an isomorphism from G to H.

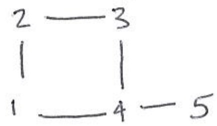
$\deg(f(u)) = \deg(u)$

deg seq $\boxed{2, 2, 2, 3, 3}$ $\boxed{1, 2, 2, 3, 4}$

Degree Sequence of a graph G is a listing of the degrees of the vertices of G in ascending order.

Proposition 1. *If G and H are isomorphic then they have the same degree sequence.*





1, 2, 2, 2, 3 1, 2, 2, 2, 3

Not isomorphic because no Δ in the right graph.

$V(H) = \{a, b, c, d, e\}$

$E(H) = \{\{a,b\}, \{b,c\}, \{c,d\}, \{d,e\}, \{c,e\}\}$

$V(G) = \{c, d, e\}$ $E(G) = \{\{c,d\}, \{d,e\}\}$

Subgraphs. G is a subgraph of H iff $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

Proposition 2. *Let G and H be isomorphic graphs. If S is a subgraph of G then there is a graph T such that T is a subgraph of H such that S and T are isomorphic.*

$\exists$ bijective $f : V(G) \to V(H)$ s.t. f is an isomorphism. Suppose $S = (V(S), E(S))$ is a subgraph G

$T = (\{f(u) \mid u \in V(S)\}, \{\{f(u), f(v)\} \mid \{u,v\} \in E(S)\} = E(T))$

Walks, Paths, and Cycles

Walk in graph G is an alternating sequence of vertices and edges that begins with a vertex, ends with a vertex, and for any edge $e = \{u, v\}$ in the walk, one of its endpoints is just before e in the sequence and the other endpoint is just after e.

$$\text{Walk is of the form } v_0\{v_0, v_1\}v_1\{v_1, v_2\}v_2 \cdots \{v_{k-1}, v_k\}v_k.$$

The **length** of a walk is the number of edges in it. walk $a\{a,b\}b\{b,c\}c\{c,e\}e\{e,d\}d\{d,c\}c$

Path is a walk such that all vertices appearing in it are distinct. $a\{a,b\}b\{b,c\}c$ path / walk

cycle → $c\{c,d\}d\{d,e\}e\{e,c\}c$ closed walk

Closed Walk is a walk that begins and ends in the same vertex. $c\{c,d\}d\{d,c\}c$ closed walk. Not cycle

Cycle is a closed walk of length > 2 such that all vertices are distinct except the first and the last.

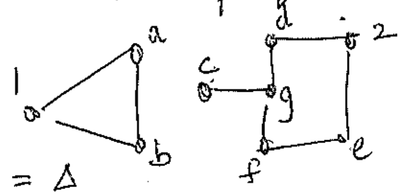
Connectivity. Vertices u and v are **connected** in graph G if there is a path that starts in u and ends in v. We denote this by $\mathsf{conn}(u, v)$. A graph G is **connected** if every pair of vertices are connected.

Proposition 3. conn *is an equivalence relation.*

Reflexive: u is walk $\mathrm{conn}(u,u)$

Symmetric: $\mathrm{conn}(u,v)$. There is walk w from u to v.
$\mathrm{rev}(w)$ is a walk from v to u $\Rightarrow$ $\mathrm{conn}(v,u)$

Transitive: $\exists$ walk p from u to v and q from v to w.
Then $p \wedge q$ is walk from u to w.



$[1]_{conn} = \{1, a, b\}$

$[2]_{conn} = \{2, c, d, e, f, g\}$

$[1]_{conn} = \triangle$

$[2]_{conn} = -\square$

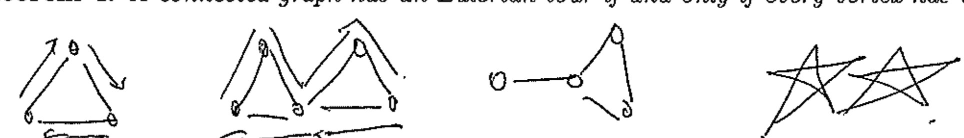
Connected Components. Equivalence classes of conn are the **connected components** of a graph G.

G is connected $\Rightarrow$ G has one connected component.

Special Walks and Tours

Eulerian Tour of G is a closed walk that includes every edge exactly once.

Theorem 4. *A connected graph has an Eulerian tour if and only if every vertex has an even degree.*



Eulerian   Eulerian   Not Eulerian

Hamiltonian Cycle of G is a cycle that visits every vertex in G exactly once.