Lecture 17: Directed Acyclic Graphs (DAGs) and Partial Orders

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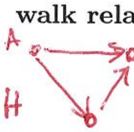
Recap

- A digraph G consists of nonempty set $V(G)$ of vertices (or nodes) and a set $E(G)$ of edges.
- A walk is an alternating sequence of vertices and edges that begins with a vertex, ends with a vertex, and such that for every edge (u, v) in the walk, u is the element just before the edge, and v is the element just after the edge in the sequence.
- A paths is a walk, where each vertex in the sequence is distinct.
- A closed walk is a walk that starts and ends in the same vertex.
- A cycle is a closed walk of length > 0 where all vertices except the first and last vertex are distinct.
- A graph G with $V(G) = \{v_0, v_1, \dots v_{n-1}\}$ can be represented by a matrix A_G where$(A_G)_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and is 0 otherwise.

Walk Relations. For a digraph $G = (V(G), E(G))$, we define a couple of binary relations on $V(G)$ — a **walk relation** G^*, and a **positive walk relation** G^+. These are defined as follows.



DAG

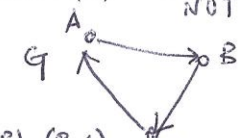
$(u, v) \in G^*$ IFF there is a walk from u to v

$(u, v) \in G^+$ IFF there is a walk of length > 0 from u to v

$H^* = \{(A,A), (A,B), (A,C), (B,B), (C,C), (C,B)\}$

$H^+ = \{(A,B), (A,C), (C,B)\}$

NOT DAG



$E = \{(A,B), (B,C), (C,A)\}$

$G^* = \{A,B,C\} \times \{A,B,C\}$

$G^+ = \{A,B,C\} \times \{A,B,C\}$

Directed Acyclic Graphs (DAGs) Digraphs with no cycles.

Proposition 1. *Every DAG has a vertex v with* $\text{indeg}(v) = 0$.

Assume $G = (V(G), E(G))$ is DAG.

Start: Pick some vertex $v \in V(G)$. If $\text{indeg}(v) = 0$ then done

O.W: $\exists$ some vertex u s.t $(u,v) \in E(G)$. Pick u.

If $\text{indeg}(u) = 0$. then done.

O.W: Repeat.

Since G is DAG, the process will stop in $|V(G)|$

Topological Sort of a digraph G is a list of all vertices such that each if $(u, v) \in E(G)$ then u appears before v in the list.

Theorem 2. *Every DAG has a topological sort.*

Pick a vertex v s.t $\text{indeg}(v) = 0$

Remove v from G along with all its outgoing edges $G - \{v\}$

$G - \{v\}$ is a DAG. Repeat.

Recap about Relations

- $R \subseteq A \times A$ is *reflexive* iff for all $a \in A$, $(a, a) \in R$.
- $R \subseteq A \times A$ is *symmetric* iff for all $a, b \in A$, $(a, b) \in R$ IMPLIES $(b, a) \in R$.
- $R \subseteq A \times A$ is *transitive* iff for all $a, b, c \in A$, $((a, b) \in R$ AND $(b, c) \in R)$ IMPLIES $(a, c) \in R$.

Proposition 3. *For a digraph G, the relations G^+ and G^* are transitive.*

Irreflexive. $R \subseteq A \times A$ is irreflexive iff for every $a \in A$, $(a, a) \notin R$.

Example: Which of the following relations on $\mathbb{N}$ is irreflexive? (a) $R = \emptyset$ I (b) $R = \{(0,0)\}$ X (c) $R = \mathbb{N} \times \mathbb{N}$ X

Proposition 4. *If G is a DAG then G^+ is irreflexive.*
Assume for contradiction G^+ is not irreflexive (and G is DAG). $\exists v$ s.t $(v,v) \in G^+$ $\Rightarrow$ G has cycle.

Strict Partial Orders. A relation $\prec A \times A$ that is transitive and irreflexive.

Examples: Standard ordering on natural numbers. $<$ — strict P.O.
(strict) Subset ordering on sets. $A \subsetneq B$ iff $A \subseteq B$ and $A \neq B$ a strict P.O.

Theorem 5. *A relation R is a strict partial order iff R is the positive walk relation for from DAG.*
($\Leftarrow$) R is G^+ for som DAG G. Then G^+ is strict P.O.

Asymmetric. $R \subseteq A \times A$ is *asymmetric* iff for every $a, b \in A$, $(a, b) \in R$ IMPLIES $(b, a) \notin R$.

Proposition 6. *If R is a strict partial order then it is asymmetric.*

Assume R is a strict partial order.
Assume (for contradiction) that R is not asymmetric.
Then $\exists (a,b)$ s.t $(a,b) \in R$ and $(b,a) \in R$.
Since R is transitive this means $(a,a) \in R$.
But R is irreflexive Contradiction.

Antisymmetric. $R \subseteq A \times A$ is *antisymmetric* iff for all a, b such that $a \neq b$, $(a, b) \in R$ IMPLIES$(b, a) \notin R$.

(Weak) Partial Order. $\preceq A \times A$ is a partial order on A iff it is reflexive, transitive, and antisymmetric.

Theorem 7. *R is a partial order iff it is the walk relation of a DAG.*