Lecture 12: Invariant Principle

Date: September 25, 2019.

Euclid's Algorithm

To compute $\gcd(a, b)$, we can assume WLOG a, b are positive, and $a \geq b$.

```
gcd(a,b)
    x = a; y = b;
    while (y > 0)
        r = rem(x,y)
        x = y
        y = r
    return x
```

State : $\mathbb{N} \times \mathbb{N}$

Transition : $(n, m) \to (m, \text{rem}(n, m))$

$(252, 198) \to (198, 54) \to (54, 36) \to (36, 18) \to (18, 0)$ $< y$

$(x, y) \to (y, \text{rem}(x, y)) \to (\text{rem}(x, y), \text{rem}(y, \text{rem}(x, y)))$ $< y$

State Machines. Binary relation, called *transitions*, on a set, called *states*.

Execution. A possible sequence of steps the machine might take, i.e., sequence of states beginning with the start state, and successive states in the sequence are related by the transition relation.

Reachable States. A state that appears in some execution.

Preserved Invariant. A predicate P on states such that whenever $P(q)$ holds and $q \longrightarrow r$ then $P(r)$ holds.

Theorem 1 (Invariant Principle, Floyd). *If a preserved invariant holds for the start state then it is true for all reachable states.*

$Q(n)$: The preserved invariant holds in any state reached in n steps

Base Case : $Q(0)$: Preserved invariant holds in start state

Ind Step : Consider any state q reached in n steps.

$\exists s$. s is reached in $n-1$ steps and $s \to q$

Ind hyp : $P(s)$ and because P is preserved invariant, $P(q)$

Proposition 2. *Preserved invariant of GCD algorithm starting from state (a, b) is that* $P(x, y)$: $\gcd(x, y) = \gcd(a, b)$.

Assume $P(x, y)$ and $(x, y) \longrightarrow (x', y')$

$P(x, y)$: $\gcd(x, y) = \gcd(a, b)$

$(x, y) \to (x', y')$: $x' = y$, $y' = \text{rem}(x, y)$

Prop : $\forall x, y \quad \gcd(x, y) = \gcd(y, \text{rem}(x, y))$

$\gcd(x', y') = \gcd(x, y) = \gcd(a, b) \Rightarrow P(x', y')$

Theorem 3. *When Euclid's algorithm halts, it correctly outputs the GCD of its inputs.*

Observe : preserved invariant holds in start state

Invariant principle : preserved invariant holds in all reachable states. in particular P holds in last state $(k, 0)$

$k = \gcd(k, 0) = \gcd(a, b)$

Termination : Because second component decreases every 2 steps

Fast Exponentiation

For $a \in \mathbb{R}$ and $b \in \mathbb{N}$, the goal is to compute a^b.

```
FastExp(a,b)
    x = a; y = 1; z = b;
    while (z ≠ 0)
        r = rem(z,2)
        z = qcnt(z,2)
        if (r = 1) then y = xy
        x = x*x
    return y
```

States: $(\mathbb{R}, \mathbb{R}, \mathbb{N})$

Trans: $(x, y, z) \to (x^2, xy, \lfloor \frac{z}{2} \rfloor)$ if z is odd

$(x, y, z) \to (x^2, y, \lfloor \frac{z}{2} \rfloor)$ if z is even

Proposition 4. *Preserved invariant for fast exponentiation is* $P(x, y, z)$: $z \in \mathbb{N}$ AND $yx^z = a^b$.

Assume $P(x, y, z)$ and $(x, y, z) \to (x_1, y_1, z_1)$

Case z is even: $z = 2k$. $x_1 = x^2$, $y_1 = y$, $z_1 = k$

$y_1 x_1^{z_1} = y(x \cdot x)^{z/2} = y(x^2)^{z/2} = yx^z = a^b$

Case z is odd: $z = 2k+1$. $x_1 = x^2$, $y_1 = xy$, $z_1 = k = \frac{z-1}{2}$.

$y_1 x_1^{z_1} = (xy)(x \cdot x)^{\frac{z-1}{2}} = x \cdot y (x^2)^{\frac{z-1}{2}} = xy x^{z-1} = y \cdot x^{1+z-1} = yx^z = a^b$.

Start State: $yx^z = 1 \cdot a^b = a^b$

Theorem 5 (Partial Correctness). *When the algorithm halts, the value returned is* a^b.

When algorithm Terminates, the preserved invariant holds.

$P(x, y, 0)$: $y \cdot x^0 = y = a^b$.

Extended GCD Algorithm

Theorem 6 (Bézout). *For any integers* a, b, $\gcd(a, b)$ *is a linear combination of* a *and* b, *i.e., there are integers* s, t *such that* $\gcd(a, b) = sa + tb$.

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```