1. Succinct Notes - Logic

Madhusudan Parthasarathy

September 1, 2017

1 Propositional Logic

Propositions: statements that are either true or false.

Syntax for propositional logic:

Fix a set of propositional variables $\mathcal{P}$. Then,

$$\varphi, \varphi' ::= \quad T \mid F \mid p \mid \neg\varphi \mid \varphi \vee \varphi' \mid \varphi \wedge \varphi'$$

where $p \in \mathcal{P}$, are the set of propositional formulas over $\mathcal{P}$.[1]

In other words, the set of formulas over $\mathcal{P}$ is the *smallest* set $\mathcal{F}$ that satisfies the following conditions:

- T and F belong to $\mathcal{F}$.
- p belongs to $\mathcal{F}$, for every p in $\mathcal{P}$.
- If φ belongs to $\mathcal{F}$, then $\neg\varphi$ belongs to $\mathcal{F}$ as well.
- If φ_1 and φ_2 belong to $\mathcal{F}$, then $\varphi_1 \vee \varphi_2$ and $\varphi_1 \wedge \varphi_2$ also belong to $\mathcal{F}$.

Semantics for propositional logic:

Let $v : \mathcal{P} \to \{\mathit{true}, \mathit{false}\}$ be any valuation. Then, under v, formulas evaluate as follows:

- T evaluates to *true*.
- F evaluates to *false*.
- p evaluates to $v(p)$, for any $p \in \mathcal{P}$.
- $\neg\varphi$ evaluates to *true* if φ evaluates to *false*, and evaluates to *false* otherwise.
- If φ_1 evaluates to *true* or φ_2 evaluates to *true*, then $\varphi_1 \vee \varphi_2$ evaluates to *true*; otherwise, it evaluates to *false*.
- If φ_1 evaluates to *true* and φ_2 evaluates to *true*, then $\varphi_1 \wedge \varphi_2$ evaluates to *true*; otherwise, it evaluates to *false*.

[1]This kind of notation is a common way of specifying the grammar for a language. You should learn how to read such notation as it's quite useful in computer science. See `https://en.wikipedia.org/wiki/Backu-Naur_form` for more information.

Some shorthand operators:

- $\alpha \Rightarrow \beta$ is defined to be $(\neg\alpha) \vee \beta$. This is read as "α implies β".
- $\alpha \Leftrightarrow \beta$ is defined to be $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$. This is read as "$\alpha$ iff β".
- α *xor* β is defined to be $(\alpha \wedge (\neg\beta)) \vee (\beta \wedge (\neg\alpha))$.

Two formulas α and β are equivalent (denoted $\alpha \equiv \beta$) iff under *all valuations*, α evaluates to the same value as β does.

Some equivalences: (below, α, β, γ are arbitrary formulas)

- Double negation: $\neg(\neg\alpha)$ is equivalent to α.
- Distributivity of $\wedge$ over $\vee$: $\alpha \wedge (\beta \vee \gamma)$ is equivalent to $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$.
- Distributivity of $\vee$ over $\wedge$: $\alpha \vee (\beta \wedge \gamma)$ is equivalent to $(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$.
- DeMorgan's laws:
  $\neg(\alpha \wedge \beta)$ is equivalent to $(\neg\alpha) \vee (\neg\beta)$.
  $\neg(\alpha \vee \beta)$ is equivalent to $(\neg\alpha) \wedge (\neg\beta)$.
- "Contrapositivity": $\alpha \Rightarrow \beta$ is equivalent to $(\neg\beta) \Rightarrow (\neg\alpha)$
- "Contradictability": α is equivalent to $(\neg\alpha) \Rightarrow F$
- "Negating Implications": $\neg(\alpha \Rightarrow \beta)$ is equivalent to $\alpha \wedge (\neg\beta)$.
  The only way an implication doesn't hold is that the premise of the implication holds and the consequence does not hold.

2 First-order Logic or Predicate Logic

Fix a universe U.
Fix some constants, $c_1, \ldots c_r$ in U.
Fix functions $f_1, \ldots, f_n$ of any arity from U to U, i.e., each function $f_i : U^k \rightarrow U$ for some arity k.
Fix relations $R_1, \ldots, R_m$ of any arity over U, i.e., each relation $R_i \subseteq U^k$, for some arity k.
(We can actually have even an infinite number of constants, functions, and relations.)

Then first-order logic is defined by first forming terms (using constants and functions of terms, perhaps many times) and then formulas using these terms (through relations, equality, Boolean combinations, and quantification).

First-order logic has the following syntax.

First, the set of *terms* are defined as:

$$t, t' :: - \quad c_i \mid f_i(t_1, \ldots t_k)$$

And the set of FO formulas are defined as:

$$\varphi, \varphi' :: - T \mid F \mid t_1 = t_2 \mid R_i(t_1, \ldots, t_k) \mid \neg\varphi \mid \varphi \vee \varphi' \mid \varphi \wedge \varphi' \mid \exists x.\ \varphi \mid \forall x.\ \varphi$$

Semantics is the natural semantics. Functions are evaluated to get elements in the universe. Relations give you true or false, and are combined using Boolean operators. The operator $\forall x.\varphi$ is interpreted to be "for every value x can take over U, φ holds". And the operator $\exists x$ is interpreted to be "there exists some value x can take over U such that φ holds."

Example:
Consider the universe $\mathbb{N}$. Let $0, 1, 2$ be constants. Let the functions be the usual functions $+ : \mathbb{N}^2 \Rightarrow \mathbb{N}$ and $* : \mathbb{N}^2 \Rightarrow \mathbb{N}$. Let the relations be the usual relations $\leq \subseteq \mathbb{N}^2$, $< \subseteq \mathbb{N}^2$, and $prime \subseteq \mathbb{N}$.

Then, the following are formulas with their meanings:

- $\forall x.\ 0 \leq x$, which says that all numbers are greater than or equal to zero. This happens to be a true statement.
- $\exists x.\ x * x = 2$, which says there is a square root of 2. This is a false statement.
- $\forall x.\ \exists y.\ x < y$, which says that for any number x, there is a number y that is greater than it. This happens to be true.
- $\forall x.\ \exists y. x + y > 2 * x$, which says that for every x, there is a y such that $x + y$ is greater than $2x$. This happens to be true (since for any x, we can always choose y to be, say, $x + 1$).
- $\exists x.\ \big(x^2 = 0 \wedge \forall y.(y^2 = 0 \Rightarrow x = y)\big)$, which says that is a unique integer x such that $x^2 = 0$.
- $\forall x.\ \exists y.(x < y \wedge prime(y))$, which says that there are infinitely many primes. This happens to be true.
- $\forall x.\ \exists y.(x < y \wedge prime(y) \wedge prime(y + 2))$, which says that there are infinitely many twin primes. This is the twin-prime conjecture. We don't know whether it's true or false!

Learn to write (or look at) any statement that you want to prove using such a logical notation.
E.g., "Any connected graph has a spanning tree": For every graph G, if G is connected, then there exists a tree T such that T is a spanning tree of G.
E.g., "Every planar graph has a 4-coloring": For every graph G, if G is planar, then there exists a 4-coloring of G.
E.g., "Every even integer greater than 2 is a sum of two primes": For every integer n, if $n > 2$ and n is even, then there exists two integers p and q such that p and q are prime and $n = p + q$.

Some useful equivalences:

- Pushing negation in:
  $\neg(\forall x.\varphi(x)) \equiv \exists x.\neg\varphi(x)$

  $\neg(\exists x.\varphi(x)) \equiv \forall x.\neg\varphi(x)$
- Contrapositivity:
  $\forall x.(\alpha \Rightarrow \beta) \quad \equiv \forall x.((\neg\beta) \Rightarrow (\neg\alpha))$

  $\exists x.(\alpha \Rightarrow \beta) \quad \equiv \exists x.((\neg\beta) \Rightarrow (\neg\alpha))$
- Forall-over-conjunction and Exists-over-disjunction:
  $\forall x.\ (\alpha \wedge \beta) \quad \equiv (\forall x.\alpha) \wedge (\forall x.\beta)$

  $\exists x.\ (\alpha \vee \beta) \quad \equiv (\exists x.\alpha) \vee (\forall x.\beta)$

  Note that the other variants of the above do not hold:
  $\forall x.\ (\alpha \vee \beta) \quad \not\equiv (\forall x.\alpha) \vee (\forall x.\beta)$

  For example, over natural numbers, consider $\alpha(x)$ saying x is even and $\beta(x)$ saying x is odd. Then $\forall x.\ (\alpha \vee \beta)$ is true but both $\forall x.\alpha$ and $\forall x.\beta$ are false, and hence $(\forall x.\alpha) \vee (\forall x.\beta)$ is false.

  Similarly, $\exists x.\ (\alpha \wedge \beta) \quad \not\equiv (\exists x.\alpha) \wedge (\forall x.\beta)$

  For example, over natural numbers, consider $\alpha(x)$ saying x is even and $\beta(x)$ saying x is odd. Then $\exists x.\ (\alpha \wedge \beta)$ is false but both $\exists x.\alpha$ is true and $\exists x.\beta$ is true, and hence $(\exists x.\alpha) \wedge (\exists x.\beta)$ is true.