CS 173: Induction

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1 Induction

This chapter covers mathematical induction, and is an alternative resource to the one in Fleck's textbook. Read both. Try to follow the induction proof outline given in this draft.

In mathematical induction, the basic setting is when we want to prove a universally quantified statement of the form $\forall n \in \mathbb{N}.P(n)$, where P is any property parameterized by n.

For example, if we want to prove that for every $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, then we could see this claim as proving $\forall n \in \mathbb{N}.P(n)$, where $P(n) : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

In order to prove such a claim $\forall n.P(n)$, a proof by induction would typically show the following:

- Prove that the claim is true when $n = 0$, i.e., show $P(0)$ is true.
- For any arbitrary integer $k > 0$, we *assume* that P holds for all smaller numbers $i = 0, \ldots, k-1$, and then prove that P holds for k.
  In other words, for any arbitrary integer $k > 0$, we assume that $P(0), \ldots, P(k-1)$ hold, and prove $P(k)$.

It turns out (see below) that if we argue the above, then that constitutes a valid proof that $P(n)$ holds for every $n \in \mathbb{N}$.

The above argument/proof can be succinctly captured as the logical formula:

$$\boxed{P(0) \ \wedge \ \forall k > 0. \left(\bigwedge_{i=0\ldots k-1} P(i)\right) \Rightarrow P(k)}$$

The informal reason why proving the above proves $P(n)$, for all n, is as follows. First $P(0)$ is true. Next, since $P(0)$ is true, and since the above shows (when $k = 1$) that *if* $P(0)$ is true then $P(1)$ is true, we know that $P(1)$ must be true. And next, since we know that $P(0)$ and $P(1)$ are true, and since we have shown (when $k = 2$) that this must imply $P(3)$ is true, we know $P(3)$ is true. We can go on like this ad infinitum to show that for any integer k, $P(k)$ must be true. This argument is, strictly speaking, not a formal proof, because this argument is infinitely long; we will give a more formal proof later below.

In the induction argument above, the first conjunct $P(0)$ is often called the *"base case"* (since it covers the simplest case on which the induction builds on). The second part of the proof $\forall k > 0. \left(\bigwedge_{i=0\ldots k-1} P(i)\right) \Rightarrow P(k)$ is called the *"induction step."*

Note that the induction step is a universally quantified statement on k. Hence the proof of the induction step often starts with "Let $k > 0$ be an arbitrary positive number.". The left-hand side of the implication in the induction step, which assumes $P(i)$ holds, for each i smaller than k, is called the *"induction hypothesis"*. We often label this as such and assume it, and then continue on to prove that $P(k)$ holds. Note that the induction hypothesis depends on k.

Now let us look at an example of a proof by induction.

Claim: For any $n \in \mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Proof. The claim is $\forall n.P(n)$, where $P(n) : \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.
We will prove the claim by induction on n.

Base-case: When $n = 0$, $\sum_{i=0}^{0} i = 0$ and $\frac{n(n+1)}{2} = 0$. Hence the claim holds when $n = 0$.

Induction step:
Let $k > 0$ be an arbitrary positive number.
Assume the **Induction Hypothesis**: For every $0 \leq j < k$, $\sum_{i=0}^{j} i = \frac{j(j+1)}{2}$.
Now let us prove that $P(k)$ holds, i.e., let us prove that $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

$\sum_{i=0}^{k} i = \sum_{i=0}^{k-1} i \ + \ k$.

Now, by the induction hypothesis, since $k-1 < k$, we know $\sum_{i=0}^{k-1} i = \frac{(k-1)k}{2}$.

Hence $\sum_{i=0}^{k} i = \frac{(k-1)k}{2} \ + \ k = \frac{k^2-k+2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2}$, which proves the claim for k.

Hence, by induction, we have proved the claim. □

Aside: Note that even though the induction hypothesis assumed P held for all values smaller than k, in the proof we used only the assumption that P holds for $k-1$. Of course, we we don't *have to* use all the assumptions that we have made in order to prove $P(k)$— we can use any of it (or none of it). Some textbooks introduce another kind of induction proof (called *weak induction*) where the inductive hypothesis itself assumes only that $P(k-1)$ holds. This restricted form is quite unnecessary and serves no purpose; we may as well assume the stronger statement that P holds on all values smaller than k. Also, when we move to problems that involve induction over more complex structures (such as trees), it is almost never the case that the weak induction form is useful. Hence we discourage the *weak induction* form entirely— never use that in this course (and you could lose points if you do).

1.1 Structure of a proof by induction

Note the structure of the above proof by induction. It's good style to follow such a structure:

- First, identify the property P and parameter n carefully such that the claim that you want to prove is $\forall n.P(n)$. You can even write down P using just English, if that's what's appropriate. But it's good to identify P.
- Say precisely what variable you are proving the induction on.
- Label the base-case(s) clearly, and prove it.
- Label the induction step part of the proof clearly.
- Your induction step should start with taking an arbitrarily large integer, say k (in this case $k > 0$); the range should cover all numbers, save the ones covered by the base case.
- State the induction hypothesis clearly. The induction hypothesis will assume the claim holds for all smaller values than the one that you are trying to prove. Note that the induction hypothesis hence depends on k.
- Finally, proceed to prove the claim for the appropriate value, i.e., prove $P(k)$.
- Have a concluding statement saying that the proof by induction proves the claim you originally set out to prove.

1.2 Different Forms of Inductive Proof

The simplest form of induction is as the logical formula:

$$\forall k \geq 0. \left(\bigwedge_{i=0 \ldots k-1} P(i)\right) \Rightarrow P(k)$$

The above simply says that for every $n \in \mathbb{N}$, $P(n)$ holds assuming P holds for strictly smaller values than n. Note that there is no explicit "base case" evident in the formula above. However, note that k ranges over all numbers (including 0), and when $k = 0$, the left-hand side of the implication is an empty conjunction, which technically evaluates to just *true*. Hence, we will be proving $P(0)$ without any assumptions, which is precisely what we call the base case. Pulling out the case when $k = 0$ gives the formula that we use:

$$P(0) \ \wedge \ \forall k > 0. \left(\bigwedge_{i=0 \ldots k-1} P(i)\right) \Rightarrow P(k)$$

There are several other mild variations of proofs by induction.

First, sometimes people find it convenient to prove $P(k+1)$ instead of $P(k)$ in the induction step. The important thing to remember is that the inductive hypothesis always is the assumption that the claim holds for strictly smaller values; hence when we want to show $P(k+1)$, we would assume $P(0), \ldots, P(k)$ hold. More formally, such an argument is captured by the logical formula:

$$P(0) \wedge \forall k \geq 0. (\bigwedge_{i=0 \ldots k} P(i)) \Rightarrow P(k+1)$$

(Note that the range of k includes 0, since the induction step should prove $P(1)$.)

I would actually discourage you to use this form— it's not really more convenient, and it encourages bad ways of thinking. The right way of thinking about induction is not that you go "from k to $k+1$", but rather that you "prove for k assuming it holds for smaller values of k". But, of course, the argument is right and an acceptable form in this course.

When proving a claim that is more naturally stated as $\forall n > 3. P(n)$, where the claim holds for all natural numbers except for the first few, a proof by induction would, of course, use a base case that tackles the first value(s) on n for which the proof is necessary (in this case $n = 4$), and then prove the induction step, where the induction hypothesis assumes the claim holds on the appropriate range of smaller than k. So an induction argument for proving this may look like:

$$P(4) \wedge \forall k > 4. \left(\bigwedge_{i=4 \ldots k-1} P(i)\right) \Rightarrow P(k)$$

Finally, it's sometimes the case that several small values of n require a different proof, before we can do a more uniform proof for larger values of

n. In this setting, it's common to prove several (but finitely many) base-cases, and then prove the induction step. So an inductive argument can be captured along the lines of:

$$P(0) \wedge P(1) \wedge P(2) \wedge \forall k > 2. \left(\bigwedge_{i=0 \ldots k-1} P(i)\right) \Rightarrow P(k)$$

Here, we prove P for the first three natural numbers explicitly, and then prove the claim for the rest using the inductive step. The above also proves $P(n)$ for every $n \in \mathbb{N}$.

1.3 Why is induction valid?

Let us now return to why proofs by induction are correct. Let us again assume that we want to prove $\forall n \in \mathbb{N}.\ P(n)$.

Let us say that we prove the following:

$$P(0) \ \wedge \ \forall k > 0. \left(\bigwedge_{i=0 \ldots k-1} P(i)\right) \Rightarrow P(k)$$

Why does this prove $\forall n. P(n)$?

Here is a simple proof (using proof by contradiction).

Proof by contradiction: Assume that $\forall n. P(n)$ is *not* true. Then there must be at least one counter-example to the claim. Let C be the set of all counter-examples to the claim, i.e., $C = \{i \in \mathbb{N} \mid \neg P(i)\}$, and C is non-empty. We know that every non-empty finite subset of natural numbers has a *least* number. Since C is a non-empty subset of natural numbers, there must be a smallest number in C, i.e., there must be a smallest counter-example. Let c be the smallest counter-example. c cannot be 0 since we have proved that $P(0)$ holds. So $c > 0$. Note that we have proved $\left(\bigwedge_{i=0 \ldots c-1} P(i)\right) \Rightarrow P(c)$ (which is the case when $k = c$). The contrapositive of this is $\neg P(c) \Rightarrow \left(\bigvee_{i=0 \ldots c-1} \neg P(i)\right)$. Hence we have proved that if there is a counter-example larger than 0, then there is a *smaller* counterexample. Hence there is a counter-example smaller than c, which violates our assumption that c is the smallest counter-example, hence showing a contradiction. Hence the only possibility is that our assumption that there is a counter-example is wrong. Hence there is no counter-example. Hence $\forall n. P(n)$.________________QED

If you are uncomfortable with the above proof by contradiction (which is covered only later in this course), you can also see soundness of induction using the following proof by contrapositive.

Proof by contrapositive: We need to show that

$$\left[P(0) \wedge \forall k > 0. (\bigwedge_{i=1}^{k-1} P(i)) \Rightarrow P(k) \right] \Rightarrow \forall n \in \mathbb{N}.\ P(n)$$

The contrapositive is:

$$\exists n \in \mathbb{N}.\ \neg P(n) \Rightarrow \left[\neg P(0) \vee \exists k > 0.(\bigwedge_{i=1}^{k-1} P(i) \wedge \neg P(k) \right]$$

Let's assume the left-hand side holds, i.e., assume $\exists n \in \mathbb{N}. \neg P(n)$. Let $C = \{j \in \mathbb{N} \mid \neg P(j)\}$, Then C is a nonempty set of natural numbers; hence there must be a least number in C. Let this least number be n. So $P(n)$ does not hold, and for every $i < n$, $P(i)$ holds.

Case 1: $n = 0$**.** In this case $\neg P(0)$ holds, and hence the right-hand side holds.

Case 2: $n > 0$**:** Choose $k = n$. Then $P(i)$ holds for every $i < k$. And $\neg P(k)$ holds. Hence $\exists k > 0.(\bigwedge_{i=1}^{k-1} P(i) \wedge \neg P(k))$ holds.________________QED

Notice that in the above proofs, the crucial point was to take the set of all counter-examples and use the fact that this set must have a *smallest* element. This property holds for natural numbers (any nonempty set of natural numbers has a smallest element). It turns out that induction is valid on *any set* with an ordering that has the above property, and this property of relations is called *well-founded relations.*

This property *does not hold* for the set of integers under the $\leq$ ordering (there are nonempty subsets of integers that do not have a smallest element— for example the set of all odd integers). So doing induction on integers directly wouldn't be correct. However, we could rephrase a claim on integers as a claim on natural numbers, and then use induction. For instance, assume that we want to prove that $\forall m \in \mathbb{Z}.\ Q(m)$ holds. We could then phrase this equivalently as $\forall n \in \mathbb{N}.\ (Q(n) \wedge Q(-n))$. This new phrasing is of the form $\forall n \in \mathbb{N}.\ P(n)$ (where $P(n) \equiv Q(n) \wedge Q(-n)$), and we can prove this by induction on n.

In general, when you are working with other structures, and you want to prove that a property holds for all structures, it is useful to associate a unique natural number with every structure, and then prove the property by induction on this natural number. For example, if you are doing induction on trees, you can associate each tree to the number of nodes in it, and do induction on the number of nodes in the tree. Or if you have a chocolate bar that you are breaking up into parts, you could do induction on the number of square pieces in the chocolate bar. Or if you are doing induction to prove a property on all $2^n \times 2^n$ grids, then you could use n as the parameter to induct on. Make sure that the variable you are inducting on is natural numbers (or a subset of natural numbers).