CS 173 Discrete Structures
Fall 2017: Homework 5 Solutions

Problem 1. A feckless tree is a 3-ary rooted tree where nodes are labeled with pairs of integers and that satisfies the following properties:

1. If n is a leaf, then n is labeled $(4, 7)$ or $(7, 12)$.
2. No internal node has exactly two children
3. If an internal node n has one child labeled (p, q), then n is labeled $(p + 3, q + 5)$.
4. If an internal node n has three children labeled (p, q), (r, s) and (t, u), then n is labeled $(2(p + r + t) + 1, 2(q + s + u))$.

Prove that in any feckless tree, if the root is labeled (p, q), then $3q = 5p+1$.

Solution: We will prove by induction on h, $h \geq 0$, that $P(h)$ holds, where $P(h)$: In any feckless tree of height h, if the root is labeled (p, q), then $3q = 5p + 1$.

Base-case: When $h = 0$, a feckless tree has a single node, which is both the root and a leaf, and its label is hence either $(4, 7)$ or $(7, 12)$.
Since $3 \cdot 7 = 5 \cdot 4 + 1$ and $3 \cdot 12 = 5 \cdot 7 + 1$, in either case the label of the root satisfies the claim.

Induction step:
Let $h \in \mathbb{N}$, $h > 0$.
Let T be a feckless tree and let r be its root.

We assume the **induction hypothesis**:

> *For any $1 < j < h$, in any feckless tree of height j, if the label of the root is (p, q), then $3q = 5p + 1$.*

Since the height of the tree is greater than 0, r has at least one child.
Since T is a feckless tree, r has either one child or 3 children.

Case 1: r has one child:
Let the label of this child be (u, v).
Then the label of r is $(u + 3, v + 5)$.
Consider the subtree rooted at this child, T'.
Notice that all the conditions of a feckless tree are met by T' (leaves are labeled $(4, 7)$ or $(7, 12)$, and labels on internal nodes of T' are related to the labels of their children in the way required, since T' is a subtree rooted at a child of r.)
Since the height of T' is less than T and T' is a feckless tree, by the induction hypothesis, $3v = 5u + 1$.
Hence $3 \cdot (v + 5) = 3v + 15 = 5u + 1 + 15 = 5(u + 3) + 1$.

Hence the label of the root satisfies the property in the claim: $3 \cdot (v+5) = 5 \cdot (u+3) + 1$.

Case 2: r has three children:
Let the labels of children be (p, q), (r, s), and (t, u).
Then the label of r is $(2(p+r+t)+1, 2(q+s+u))$.
Consider the subtrees rooted at the children of r.
Notice that all the conditions of a feckless tree are met by these subtrees, and hence they are also feckless trees. Since the height of all of these subtrees are less than that of T and they are feckless, by the induction hypothesis, $3q = 5p+1$, $3s = 5r+1$, $3u = 5t+1$.

Turning to the label of r,
$3 \cdot 2(q+s+u) = 2 \cdot (3q+3s+3u) = 2 \cdot (5p+5r+5t+3) = 5 \cdot (2p+2r+2t+1)+1 = 5 \cdot (2(q+r+t)+1)+1$.
Hence the label of the root satisfies the property in the claim. □

Problem 2. For this problem, consider binary trees where nodes can have either a left child and/or a right child, or no children (in other words, children are either left or right children, and a node can have at most one left child and at most one right child).

A polarized tree is a labeled tree $T = (V, E, lab)$, where (V, E) is a binary tree and $lab : V \rightarrow \mathbb{N} \times \mathbb{N}$ is a function that gives a label consisting of a pair of natural numbers for each vertex in the tree that satisfies the following conditions:

- For any vertices u, v, with $u \neq v$, u labeled (a_1, b_1) and v labeled (a_2, b_2), if v is a descendant of u, then $a_2 < a_1$.
- For any vertex u with label (a, b), if it has a left child v_1, then for any descendant v of v_1 (including v_1 itself, of course), if v has label (a', b'), then $b' < b$.
- For any vertex u with label (a, b), if it has a right child v_1, then for any descendant v of v_1 (including v_1 itself, of course), if v has label (a', b'), then $b' > b$.

The set represented by a polarized tree is the set of labels of all its nodes.

Let S be an arbitrary nonempty finite subset of $\mathbb{N} \times \mathbb{N}$ that has unique numbers in the first component, and unique numbers in the second component. More precisely, let S satisfy the property that for any two distinct elements (a_1, b_1) and (a_2, b_2) in S, $a_1 \neq a_2$ and $b_1 \neq b_2$.

Prove that for any such set S, there is a polarized tree T such that the set represented by T is S.

Hint: Use induction on the size of S. When proving a tree representing S exists, decide what the label of the root must be, and decide what subsets *the left subtree and right subtree of the root would represent. The proof should then be easy.*

Solution:

We will prove by induction on $|S|$, the size of $|S|$, that there is a polarized tree T that represents S, for any nonempty finite set $S \subseteq \mathbb{N} \times \mathbb{N}$ that has unique numbers in the first component and the second component.[1]

Base-case: When $|S| = 1$, let (a, b) be the sole element in S. Consider a tree with only one node, which is also the root, and that is labeled (a, b). Clearly, this is a polarized tree that represents S.

Induction step:

Let S have n elements $n > 1$ with unique first components and unique second components.

Let us assume the *induction hypothesis*:
For any nonempty finite set $T \subseteq \mathbb{N} \times \mathbb{N}$ with less than n elements and with unique first and second components, there is a polarized tree that represents T.

Let (a_0, b_0) be an element in S that has the largest first component, i.e., $\forall (a, b) \in S, a \leq a_0$. (This element exists since S is nonempty; in fact it is unique since S has unique first components.) Let us construct a polarized tree that represents S with (a_0, b_0) as the label of the root.[2]

Let S_1 be the set of all elements (a, b) in S such that $b < b_0$. And let S_2 be the set of all elements (a, b) in S such that $b > b_0$.[3]

Note that $S = S_1 \cup S_2 \cup \{(a_0, b_0)\}$ and all these three sets are disjoint. (Proof: they are clearly disjoint by choice of S_1 and S_2. Also, clearly, $S_1 \cup S_2 \cup \{(a_0, b_0)\} \subseteq S$. Let's show $S \subseteq S_1 \cup S_2 \cup \{(a_0, b_0)\}$. Note that if (a, b) is in S and is different from (a_0, b_0), then $b \neq b_0$ since S has unique first components, and hence $b < a_0$ or $b > a_0$. In the former case, (a, b) is in S_1 and in the latter case it is in S_2.)

Since $(a_0, b_0) \notin S_1$ and $(a_0, b_0) \notin S_2$, we know $|S_1| < n$ and $|S_2| < n$. Also, since they are subsets of S, they also have unique first and second components. Hence, by the induction hypothesis, if S_1 is nonempty, there is a polarized tree (say T_1) that represents S_1, and if S_2 is nonempty, there a polarized tree (say T_2) that represents S_2.

[1] The footnotes here are not part of the proof. You could include them, but they are not necessary.

[2] Intuiton: clearly any tree that represents S must have (a_0, b_0) as the label of its root, because of the first condition of polarized trees.

[3] Intuition: in the polarized tree we are constructing, clearly S_1 must be the set represented by the left subtree of the root and S_2 must be the set represented by the right subtree of the root, because of the second and third conditions of polarized trees.

Construct now a tree T with a root node labeled (a, b) and, if S_1 is nonempty, let it have a left child with the subtree rooted at this child being T_1, and if S_2 is nonempty, let it have a right child with the subtree rooted at this child being T_2.

We claim T is a polarized tree representing S. Clearly the labels of the nodes of T is the set S, since $S = S_1 \cup S_2 \cup \{(a_0, b_0)\}$.

Let us prove the three properties of polarized trees for the tree T.

First, let v, labeled (a_2, b_2) be a descendent of u, labeled (a_1, b_1), $u \neq v$, in T. Then if u, v both belong to T_1, then $a_1 < a_2$ since T_1 is a polarized tree. Similarly, if u, v both belong to T_2 then $a_1 < a_2$ since T_2 is a polarized tree. The only other case is when u is the root of T and hence $a_1 = a_0$. By choice of a_0, $a_1 = a_0 < a_2$. This proves the first properties of polarized trees for T.

Second, let u, labeled (a, b) have a left child whose descendent is a node v, labeled (a', b'), in T. Then if u, v both belong to T_1, then $b' < b$ since T_1 is a polarized tree. Similarly, if u, v both belong to T_2 then $b' < b$ since T_2 is a polarized tree. The only other case is when u is the root of T (hence $b = b_0$) and v is a node of T_1. Then by choice of S_1, we know that $b' < b_0 = b$. This proves the second property of polarized trees for T.

Third,[4] let u, labeled (a, b) have a right child whose descendent is a node v, labeled (a', b'), in T. Then if u, v both belong to T_1, then $b' > b$ since T_1 is a polarized tree. Similarly, if u, v both belong to T_2 then $b' > b$ since T_2 is a polarized tree. The only other case is when u is the root of T (hence $b = b_0$) and v is a node of T_2. Then by choice of S_2, we know that $b' > b_0 = b$. This proves the third property of polarized trees for T.

We have hence shown that T is a polarized tree that represents S.[5] [6] □

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[4] We could have also said here that the third property follows by a similar argument as the second property above, as the proof is really similar. In a long proof, this is justified, but if you do this, make sure it is really similar and make sure you run it in your head and ensure it is correct.

[5] Intuition: In fact, we didn't have much of a choice in constructing T. The root was forced and the sets represented by the left and right subtrees were forced, leaving us no choice. We can actually extend the above proof to show that there is in fact only *one* polarized tree that represents S.

[6] Aside: these objects are actually called treaps— they are binary search trees on the second component and max-heaps on the first component; see `https://en.wikipedia.org/wiki/Treap`. In fact, by choosing random priorities (first component) when inserting elements into a binary search tree can give randomized binary search trees that are most likely balanced, i.e., of roughly logarithmic height.