CS 173 Discrete Structures
Fall 2017: Homework 4 Solutions

Problem 1. Show that any connected graph with more than one vertex is weakly 2-colorable.

A graph is weakly 2-colorable if every vertex can be assigned a color, red or blue, such that for any vertex u in G has *some* neighbor whose color is different than that of u.

Note: weak colorability is different from colorability.

Prove this by induction (we leave you to figure out whether to induct on the number of vertices or number of edges).

Solution:

One basic issue in this problem is that when you remove a vertex from a connected graph, the graph breaks into connected components, but some of them may have a single vertex, which makes it hard to use the induction hypothesis. There are several ways to handle this.

We give *four* proofs: three full proofs and one proof gist.

The first proof here is perhaps the simplest. It handles the issue by considering the resulting isolated vertices, and carefully constructing the coloring.

The second proof below chooses the vertex to remove carefully so that it doesn't result in any isolated vertex. However, it strengthens the hypothesis to weakly 2-color *any* graph with no isolated vertices (even those that are not connected) and hence the induction step is simpler.

The third proof also chooses the vertex to remove carefully so that it doesn't result in any isolated vertex. But does not strenthen the claim to prove, and hence the induction step is more involved.

The fourth proof (we give only an outline)avoids the entire problem by doing induction on number of edges. Intuitively, you keep removing edges till you get a tree that connects all nodes in the graph, and then the tree is easily seen to be weakly 2-colorable and that same coloring will be a weak 2-coloring of the entire graph.

Proof #1:

We will prove the claim by induction on the number of nodes, n. That is, we will prove by induction on n, that any connected graph with n nodes is weakly 2-colorable, for every $n \in \mathbb{N}$ with $n > 2$.

Base-case: $n = 2$.
The only connected graph with two vertices has the two vertices connected by an edge. We can assign one of them *red* and the other *blue*, and the graph is clearly weakly 2-colorable.

Induction step:
Let $n \in \mathbb{N}$, $n > 2$ and let G be a connected graph with n nodes.

Let us assume the *induction hypothesis*:
For any connected graph G with m vertices, where $2 \leq m < n$, G is weakly 2-colorable.

Let u be an arbitrary vertex of G. Let G' be the graph obtained by removing u and all edges incident on u from G. Let $v_1, \ldots, v_k$ be the neighbors of G (since G is connected and has more than 1 vertex, u must have at least one neighbor, i.e., $k \geq 1$).

Now consider all the connected components of G'. For every connected component of G' that has more than one vertex, we know by the induction hypothesis (since the component has less than n vertices) that it is weakly 2-colorable. Let us take the coloring of G' that uses this coloring for each such connected component, and for every other component (which has only one vertex), let us choose to color it *red*. This gives a coloring of G' (it is not necessarily a weak coloring of G').

Now, consider G. If all the neighbors of $v_1, \ldots v_k$ are colored *blue*, then color u *red*, and otherwise (i.e., if any neighbor is colored red), color u *blue*. We claim that this is a weak 2-coloring of G.

First, clearly u will have a neighbor of a different color (since $k \geq 1$).

Second, in G', notice that a connected component that has only one vertex must be a neighbor of u (if not, then in G, this vertex will have no neighbors, violating the assumption that G is connected). Hence, if there are any such vertices, these will be neighbors of u and will get colored red, and hence u will get colored blue, and hence every such neighbor will have a neighbor (namely u) that has a different color.

Third, consider a vertex that is in a connected component of G' that has more than one vertex. Then, we know the coloring for that component ensured that it had a neighbor of a different color. When we extend this coloring to G, the property is clearly maintained.

We have hence argued that every vertex of G has a neighbor of a different color, hence proving the claim.________________________QED.

Proof #2: We will prove by induction on n, where $n > 1$, the slightly stronger claim that:
$P(n)$: Any graph with n nodes that has no isolated vertex (a vertex with degree 0) is weakly 2-colorable.

Note that proving the above proves the claim that we want to prove since in any connected graph with more than one vertex has sno isolated vertex.

Base-case: $n = 2$.
The only graph with 2 vertices that has no isolated vertex is the one that has the two vertices connected by an edge. We can assign one of them *red* and the other *blue*, and then the graph is clearly weakly 2-colorable.

Induction step:
Let $n \in \mathbb{N}$, $n > 2$ and let G be a graph with n nodes and no isolated vertex.

We assume the **induction hypothesis**:

Any graph with at least two nodes and less than n nodes that has no isolated vertex is weakly 2-colorable.

Case 1: There is a vertex u in G with degree 1.
Let (u, v) be the edge incident on u.
Consider the graph G' obtained by removing u and the edge (u, v).
The graph G' has no isolated vertex (if G' has an isolated vertex, then v would have to be that vertex and the only edge incident on v would have to be (u, v), in which case v won't be connected to the rest of G').
Also, G' has $n - 1$ nodes, where $n - 1 > 1$.
Hence, by the induction hypothesis, G' is weakly colorable.
We can extend this to a weak coloring of G: if v is colored *red*, then we color u *blue*; if v is colored *blue*, we color u *red*; this is clearly a weak coloring of G.

Case 2: There is no vertex in G with degree 1.
Let u be any vertex in G.
Consider the graph G' obtained by removing the vertex u and all edges incident on u (there must be at least one, since u is not isolated) from G.
The graph G' has no isolated vertex (since all vertices had at least degree 2, and the degree of any vertex in G' can go down at most by 1 because of the removal of u). Hence, by the induction hypothesis, G' is weakly two colorable.

Now, let us extend this weak coloring of G' to get a weak coloring of G. Consider the neighbors of u in G (there must be at least one since u is not isolated), say $v_1, \ldots, v_k$. If all of them were assigned the same color (in the weak coloring of G'), then assign u the other color (if all of them were assigned red, assign blue to u; if all of them were assigned blue, then assign u red). If there are some neighbors assigned red and some assigned blue, then assign u any color (say, red). Clearly, this results in a weak coloring of G, proving the claim. QED

Proof #3: We will prove by induction on n, where $n > 1$, that:
$P(n)$: Any graph with n nodes that is connected is weakly 2-colorable.

Base-case: $n = 2$.
The only graph with 2 vertices that is connected is the one that has two vertices connected by an edge. We can assign one of them *red* and the other *blue*, and then the graph is clearly 2-colorable.

Induction step:
Let $n \in \mathbb{N}$, $n > 2$.
Let G be a connected graph with n nodes.

We assume the **induction hypothesis**:

Any connected graph with at least 2 nodes and less than n nodes is weakly 2-colorable.

Case 1: There is a vertex u in G with degree 1.
Let (u, v) be the edge incident on u.
Consider the graph G' obtained by removing u and the edge (u, v).
The graph G' is connected (since a simple path in G from any two vertices in G' could not have gone through u, since u has only one edge incident on it).
Also, G' has $n - 1$ nodes, where $n - 1 > 1$.
Hence, by the induction hypothesis, G' is weakly colorable.
We can extend this to a weak coloring of G: if v is colored *red*, then we color u *blue*; if v is colored *blue*, we color u *red*; this is clearly a weak coloring of G.

Case 2: There is no vertex in G with degree 1.
Let u be a vertex in G.
Consider the graph G' obtained by removing u and all edges incident on u (there must be at least one, since G is connected) from G.
The graph G' may not be connected. But consider all maximally connected components of this graph. None of these maximally connected components can have only one vertex (since all vertices had degree at least 2 in G, and removing u cannot decrease their degree by more than 1). Also each of the maximally connected subgraphs of G' has less than n nodes. So, using the induction hypothesis, each of the maximally connected subgraphs of G' is weakly two-colorable. Putting them together, G' itself is weakly 2-colorable.

Now, let us extend this weak coloring of G' to get a weak coloring of G. Consider the neighbors of u in G (there must be at least one), say $v_1, \ldots, v_k$. If all of them were assigned the same color (in the weak coloring of G'), then assign u the other color (if all of them were assigned red, assign blue to u; if all of them were assigned blue, then assign u red). If there are some neighbors assigned red and some assigned blue, then assign u any color (say, red). Clearly, this results in a weak coloring of G, proving the claim. QED

Proof #4:
Here is another proof outline. (Note: you need to flesh this out to a complete proof.)

We do induction on number of *edges*. The base case is handled as above (when number of edges is 1). In the induction step, consider two cases. In Case 1, consider the graph having a cycle. Remove any edge in a cycle; the graph will remain connected, and using the induction hypothesis, will have a weak 2-coloring. Adding the edge back will not destroy this. In the second case, the graph has no cycle and is connected. So it is a tree. A tree can be weakly 2-colored easily: take some node as the root and color it red; color nodes red if they are at an even distance from the root, and blue if they are at an odd-distance. (This is in fact a 2-coloring of the tree, and also a weak 2-coloring.)

Problem 2. There is a grid of prison cells of n cells by n cells, each cell holding a prisoner.

The prisoners want to plan a prison break which requires all the prisoners to act at once. They want to decide on a particular time for the breakout.

A group of r prisoners meet and decide on a time, where $r < n$. They then decide to spread the decided upon time by communicating with prisoners in neighboring cells— prisoners in immediately close cells to the north, south, west, and east. However, a prisoner will believe the time only if he/she hears from one neighbor north/south and one neighbor east/west (e.g., a prisoner would believe the time if the neighbor north and the neighbor east tell him the time; but a prisoner would not believe the time if the neighbor north and the neighbor south pnly told him the time).

Prove that for any $n > 1$ and any $r < n$, no matter where these r prisoners are, and no matter how many days elapse, there will be at least one prisoner who does NOT know the time for revolt.

You need to prove (b) by induction formally.

Hint: Show that initially there is a particular set of prisoners that do not know the time, and then show that these prisoners will never get to know the time ever, by induction.

Solution:

We will show by induction on the number of days D that elapse that there will always be one row and one column of prison cells such that the prisoners in these cells do not know the agreed upon time.

Base-case: When $D = 0$, there are only r prisoners that know the time and $r < n$. Consequently, no matter where these prisoners are, there will be at least one row that does not include these r prisoners (since $r < n$) [Pigeonhole principle]. Similarly, there will be at least one column of prison cells that does not have any of these r prisoners. Which proves the property for $D = 0$.

Induction step:
Now consider day D, where $D > 0$. We know, by induction hypothesis, that our claim holds for day $D - 1$, i.e., on the previous day, there is at least one row and one column of prison cells such that the prisoners in these cells do not know the secret time. Let this row be r and this column be c.

Now, it is easy to see that none of the prisoners in row r will get to know the secret time, since they don't have anyone in the east/west direction who know the secret. Similarly, none of the prisoners in column c will get to know the secret time, since they don't have anyone in the north/south direction who know the secret. Hence, the prisoners in row r and column c will continue not knowing the secret in day D.

The above proof proves shows there is at least one row and one column of prisoners who never get to know the secret, and hence shows that there is at least one prisoner who never gets to know the secret.

Problem 3. This problem involves *directed graphs*. A directed graph is a pair (V, E) where V is a finite set of vertices and $E \subseteq V \times V$ is a set of edges. An edge (u, v) means that there is an edge from u to v. There can be graphs that have an edge from u to v but not from v to u.

For this problem, there's nothing you need to know about directed graphs other than the above definition.

A directed graph $G = (V, E)$ is said to be *densely connected* if for every pair of vertices u and v in V, $u \neq v$, there is either and edge from u to v or from v to u, but not both.

A Hamiltonian path in a directed graph is a path (the path must respect the directionality of edges, of course) that visits *all* vertices of the graph and visits them exactly once.

Prove that every densely connected graph with more than one vertex has a Hamiltonian path.

Solution: We will prove by induction on n, $n > 1$, that $P(n)$ holds, where $P(n)$: Any densely connected graph with n vertices has a Hamiltonian path.

Base-case: When $n = 2$, the only densely connected graph over two vertices is one that has a directed edge between the two vertices. This edge by itself is a Hamiltonian path.

Induction step:
Let $n \in \mathbb{N}$, $n > 2$.
Let $G = (V, E)$ be a densely connected graph with n vertices.

We assume the **induction hypothesis**:
For any $1 < j < n$, any densely connected graph with j vertices has a Hamiltonian path.

Let v be any node in G.
Let H be the subgraph obtained by removing v from G and all edges incident on v from G.
Clearly H is a densely connected graph, since for any two vertices p, q in H, there was exactly one edge between p and q in G, which will remain in H.

Since H has $n-1$ vertices, by the induction hypothesis we know that H has a Hamiltonian path that goes through all vertices of H.
Assume this Hamiltonian path is:

$$u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_{n-1}$$

Let us now consider G. Note that for every vertex u_i, there exists either an edge from u_i to v or from v to u_i (for every i, $1 \leq i \leq n-1$)
We now consider three cases:

Case 1: There is an edge from u_{n-1} to v.
In this case, G has a Hamiltonian path

$$u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_{n-1} \rightarrow v$$

Case 2: There is an edge from v to u_1.
In this case, G has a Hamiltonian path

$$v \rightarrow u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_{n-1}$$

Case 3: Neither of the above two cases hold.
In this case, there is an edge from u_1 to v and there is an edge from v to u_{n-1}.
Let k be the *smallest* index such that there is an edge from v to u_k.
Such an index k must exist in the range $[2, n-1]$ (since there is an edge from v to u_{n-1}).
Since k is the smallest such index, it follows that there is an edge from u_{k-1} to v. (If k=2, notice that there is an edge from u_1 to v).
Then the following is a Hamiltonian path in G:

$$u_1 \rightarrow \ldots u_{k-1} \rightarrow v \rightarrow u_k \rightarrow u_{k+1} \rightarrow \ldots \rightarrow u_{n-1}$$

We have hence proved that every densely connected graph with more than one vertex has a Hamiltonian path. □