CS 173 Discrete Structures
Fall 2017: Homework 3 Solutions

1. *Equivalence Relations*:

   Prove that R is an equivalence relation on $\mathbb{Z}$.

   Let R be the following relation $\mathbb{Z}$:

   aRb iff $5 \mid (2a + 3b)$

   Prove that R is an equivalence relation.

   SOLUTION:
   We prove that R is an equivalence relation, by showing it is reflexive, symmetric, and transitive.

   Reflexive: Let $x \in \mathbb{Z}$ be an arbitrary integer.
   Then $2x + 3x = 5x$, and hence $5 \mid (2x + 3x)$. Hence xRx.

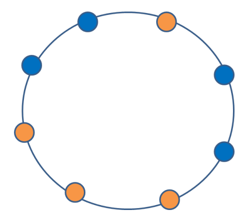
   Symmetric: Let $x, y \in \mathbb{Z}$ be arbitrary integers.
   Assume xRy. Then, by definition of R, $5 \mid 2x + 3y$.
   Let $m \in \mathbb{Z}$ such that $5m = 2x + 3y$.
   Now, $4 \cdot (2y + 3x) = 8y + 12x = (5y + 10x) + (3y + 2x) = 5(y + 2x + m)$.
   So $5 \mid (4 \cdot (2y + 3x))$.
   Since 5 is prime, and $5 \not\mid 4$, we have $5 \mid (2y + 3x)$.
   Hence yRx holds, proving R is symmetric.

   Transitive: Let $x, y, z \in \mathbb{Z}$ be arbitrary integers.
   Assume xRy and yRz.
   Then, by definition of R, $5 \mid 2x + 3y$ and $5 \mid 2y + 3z$.
   So $5 \mid (2x + 3y + 2y + 3z)$, i.e., $5 \mid (2x + 3z + 5y)$.
   So $5 \mid (2x + 3z + 5y - 5y)$ (since $5 \mid -5y$).
   So $5 \mid 2x + 3z$. Hence xRz. ______ **QED.**

2. Consider a circle with an equal number of orange stations and blue stations, say n of them each, placed on the circumference of the circle (stations do not touch each other).

   Show that you can always start from somewhere on the circle and walk clockwise around the circle such that the number of orange stations seen is always greater than or equal to the number of blue stationss seen.

   For example, if the stationss were placed in the order OBBOOOBB, as shown below,



then we can start from the 4th point, and we will see OOOBBOBB, and hence the number of orange stationss seen will always be greater than or equal to the number of blue stations seen.

Prove this by induction on n.

Hint: When reducing from a larger number of stations to smaller, try to remove two consecutive stations, and carefully choose these!

Solution:

We will prove by induction on n, $n > 0$, that $P(n)$ holds, where
$P(n)$: For any circle with n blue stations and n orange stations, there is a station on the circle where we can start so that if we go clockwise from there, then at any time, the number of orange stations we would have seen would be at least as many as the number of blue stations we have seen.

Base-case: When $n = 1$, there is a single orange station and a single blue station. We can start at the orange station, and the claim is clearly true.

Induction step:
Let $n \in \mathbb{N}$, $n > 1$.

We assume the **induction hypothesis**:
For any circle with j points, where $0 < j < n$, we can start at some station on the circle so that if we go clockwise, at any time, the number of orange stations we would have seen would be at least as many as the number of blue stations we have seen.

Consider a circle C with n orange stations and n blue stations.

Now, there must be, somewhere on this circle, an orange station followed by a blue station.
(Proof: Start from any orange station and go clockwise; you must see a blue station at some time; the first blue station we find will be a point where an orange station is followed by a blue station.) Let us call these points p and q.

Now consider the circle C' without the points p and q. This new circle has $n-1$ orange stations and $n-1$ blue stations. Hence by the induction hypothesis, there exists a station x on this circle so that if we start from x and go clockwise, the number of orange stations we would have seen would be at least as many as the number of blue stations we have seen.

Now consider C. Let us choose the same station x to start in the circle C. Note that x cannot be between p and q (since p and q were consecutive). So when we go clockwise from x, we will again always see more (or equal) orange points than blue points. To see why, note that we will first reach p, and before reaching p, we would not have seen q, and hence the property will hold till we reach p (since the property held in C'). When we reach p, the property will hold since p is colored orange. We will then immediately see q, and again the property will hold, since it held before seeing p, and we have seen one orange and one blue after it. After q, it will continue to hold, since the property held in C' and the only difference is that we have seen one more orange and blue.

Hence, we have shown by induction that $P(n)$ holds, proving the claim. $\square$