CS 173 Discrete Structures
Fall 2017: Homework 2 Solutions

1. *Induction*: For every natural number $n \in \mathbb{N}$, prove that $7 \mid 9^n - 2^n$.

   SOLUTION:
   We will prove by induction on n that $P(n) : 7 \mid 9^n - 2^n$ holds, for every $n \in \mathbb{N}$.

   <u>Base-case:</u> When $n = 0$, $9^0 - 2^0 = 1 - 1 = 0$ and $7|0$.
   Hence $7 \mid 9^n - 2^n$ when $n = 0$.

   <u>Induction step:</u>

   Let $k > 0$ be an arbitrary natural number.
   Let us assume the *induction hypothesis*:

   For every $i \in \mathbb{N}$ *such that* $0 \leq i < k$, $7 \mid 9^i - 2^i$.

   We need to show that $7 \mid 9^k - 2^k$.

   $$9^k - 2^k = 9(9^{k-1} - 2^{k-1}) + 9 \cdot 2^{k-1} - 2 \cdot 2^{k-1}$$
   $$= 9(9^{k-1} - 2^{k-1}) + 7 \cdot 2^{k-1}$$

   By the induction hypothesis, $7|(9^{k-1} - 2^{k-1})$ and hence $(9^{k-1} - 2^{k-1}) = 7r$ for some $r \in \mathbb{Z}$.

   Hence $9^k - 2^k = 7 \cdot 9 \cdot r + 7 \cdot 2^{k-1}$
   $= 7(9 \cdot r + 2^{k-1})$.

   Since $9 \cdot r + 2^{k-1} \in \mathbb{Z}$, we conclude that $7|(9^k - 2^k)$.

   Hence, we have proved by induction that $7|(9^n - 2^n)$, for every $n \in \mathbb{N}$.

   ______________________________ **QED.**

2. *Induction*:

   Two players A and B play a game where they take turns adding numbers from 1 through 8, and the first person who gets to the target of 100 wins. Assume A starts the game, and A declares a number between 1 and 8 (both inclusive). B then adds a number between 1 and 8 to this number. A does the same. And they play on in this way.

   For example, a play could go as follows:

   $A:5$ $B:8$ $A:16$ $B:17$ $A:24$ $B:30$ $A:37$ $B:45$ $A:46$
   $B:47$ $A:49$ $B:57$ $A:65$ $B:70$ $A:78$ $B:83$ $A:88$ $B:89$
   $A:90$ $B:95$ $A:100$

   and A wins.

   Note that in each move, the player must add something from 1 to 8 to the current number.

We say A has a winning strategy if there is a way for A to play such that *no matter* how B plays such that A always wins.

We want to prove that A has a winning strategy.

Prove by induction on n that if the game were to be played so that $(9n+1)$ is the target, for *any* natural number $n > 0$, then A has a winning strategy. Use this to argue that A has a winning strategy in the above game to reach 100 or in the game to reach 1000000.

SOLUTION:

Let us prove that A has a winning strategy in the game with target $9n+1$, for every natural number $n > 0$.

We will show this by induction on n.

Base-case: When $n = 1$, in the game with target $9n + 1 = 10$, A can win as follows. First A chooses 1. No matter what value B picks, the resulting number will be i where i is in the range $[2, 9]$. A can now win by declaring adding $10 - i$ to get to the target 10, since $10 - i$ is in the range $[1, 8]$.

Induction step:

Let $k > 1$ be an arbitrary natural number.
Let us assume the *induction hypothesis*:

For every $i \in \mathbb{N}$ such that $1 \leq i < k$, A has a strategy to win the game with target $9i + 1$.

We need to show that A can win the game with target $9k + 1$.

Let A first play as the game as though the target was $9(k-1)+1 = 9k-8$. Note that the rules of the game do not depend on the target, and the valid moves of player A and player B are the same till the lower target is reached.

Then, by the induction hypothesis, A has a winning strategy in this game, and hence can force the game to reach $9k-8$ ending with A having played last.

Now no matter what value B picks, the resulting number will be i where i is in the range $[9k - 8 + 1, 9k - 8 + 8]$, i.e., $[9k - 7, 9k]$. A can now win by adding $(9k + 1) - i$ to get to the target 10, since $(9k + 1) - i$ is in the range $[1, 8]$.

Hence A has a winning strategy for the game with target $9k + 1$.

Hence, we have proved by induction that A has a winning strategy for the game with target $9n + 1$, for any $n > 1$.

Consequently, A has a winning strategy for the target $100 = 9 \cdot 11 + 1$ and also for the target $1000000 = 9 \cdot 111111 + 1$.

QED.

3. *Return of the Guild of Parity Milliners*:

SOLUTION:

We will prove by induction on (discrete) time $t \in \mathbb{N}$ that after time n, the guild can maintain its rule.

Base-case: When $n = 0$, only Linus is in the guild wearing a red hat, and the code is already satisfied.

Induction step:

Let $k > 1$ be an arbitrary natural number.
Let us assume the *induction hypothesis*:

At every time $t < k$, the guild can maintain its code.

By the induction hypothesis, we know that at time $k - 1$, the guild can maintain its code. Let us consider the configuration of people wearing hats at time $k - 1$ that maintains the code.

At time $k + 1$, one of three things can happen:

- A new candidate passes the test, and either knows only one person in the guild or knows two people in the guild who do not already indirectly know each other. We had proved in HW1, that in this case, the guild can, by possibly changing hats, continue to maintain its code.
- Two people in the guild get to know each other, and the newer member gets thrown out of the guild. In this case, everyone can keep their hat, and the code will still be maintained (if any two people know each other, they will already have different colored hats).
- Nothing happens (no new candidate, no one gets to know each other in the guild). Clearly the code is maintained.

Hence the guild can always maintain its code.

□