

CS 173 Discrete Structures
Spring 2016: Homework 1 Solutions

1. *Direct Proof:*

Let u, x, y, z be integers and assume that 11 divides $(u - x + y - z)$.
Then prove that 11 divides $(1000u + 100x + 10y + z)$ as well.

An integer x is said to be divisible by 11 if there is some integer m such that $x = 11m$.

SOLUTION:

Let $u, x, y, z \in \mathbb{Z}$ and assume that $u - x + y - z$ is divisible by 11.
By the definition of “divisible by 11”, $u - x + y - z = 11m$ for some $m \in \mathbb{Z}$.
Now,

$$\begin{aligned} 1000u + 100x + 10y + z &= (1001u + 99x + 11y) - (u - x + y - z) \\ &= 11(91u + 9x + y) - 11m \\ &= 11(91u + 9x + y - m) \end{aligned}$$

Since $u, x, y, z, m \in \mathbb{Z}$, we know that $91u + 9x + y - m \in \mathbb{Z}$.
Hence $1000u + 100x + 10y + z = 11k$ for some $k \in \mathbb{Z}$.
Thus, by definition, $1000u + 100x + 10y + z$ is divisible by 11.

_____ **QED.**

2. Use *Proof by Contrapositive* to solve the following:

Let $n \in \mathbb{Z}$. Prove that if $n^2 - 6n + 3$ is even, then n is odd.

Recall the definition of odd/even: An integer m is even if there exists an integer k such that $m = 2k$. An integer m is odd if it is not even.

SOLUTION:

Let us prove the following contrapositive to the statement, which is:
For every $n \in \mathbb{Z}$, if n is even then $n^2 - 6n + 3$ is odd

Let $n \in \mathbb{Z}$ be an arbitrary integer and assume n is even.

Then there exists an integer m such that $n = 2m$.

Hence $n^2 - 6n + 3 = 4m^2 - 12m + 3 = 2(2m^2 - 6m + 1) + 1$.

Since $2m^2 - 6m + 1$ is an integer, $n^2 - 6n + 3 = 2k + 1$ for some integer k ,
and hence is odd.

(Note: If an integer x is equal to $2p + 1$ for some integer p , then it must be odd, since there cannot be any integer q such that $x = 2q$, by the Division Theorem.)

_____ **QED.**

3. Hat cases:

The Guild of Parity Milliners is a cult where members wear blue hats and red hats, and adhere to the code that no two people who know each other can wear the same colored hat.

Maricha, a member of this cult, invites Thelma, who is not part of the cult, to join the cult.

- (a) Assume Thelma does not know anyone other than Maricha in the cult. Thelma wants to wear only a red hat. Prove that after Thelma joins, there is a way for the group to maintain its code, by possibly changing hats for members.
- (b) Assume Thelma knows two people in the cult, Maricha and Subahu (and she has no preference of hat color). But Maricha does not have even an indirect connection to Subahu— i.e., Maricha doesn't know someone who knows someone who knows someone ... who knows Subahu. Prove again that after Thelma joins, there is a way for the group to maintain its code.

You can assume the club has enough hats of either color. You can assume that "knowing" is symmetric— if A knows B, then B knows A too.

SOLUTION:

- (a) Let us consider two cases:

Case 1: Maricha is currently wearing a blue hat.

In this case, Thelma can join in with a red hat, and since she has a different hat color than Maricha, who is the only person she knows, and further any other two people who knew each other already had different colored hats, we meet the code of the guild.

Case 1: Maricha is currently wearing a red hat.

Let us switch the hats of the people in the guild (before Thelma joins). That is, let the people with blue hats wear red hats, and those wearing red hats wear blue hats. Clearly, the code of the guild is still met as any two people who know each other would have had different colored hats, and will continue to have different colored hats after everyone switches colors. After this switch, Maricha is wearing a blue hat. And we already proved by Case 1 that Thelma can join while maintaining the code of the cult.

- (a) Let us consider two cases:

Case 1: Maricha and Subahu are wearing the same colored hat.

In this case, Thelma can wear a hat of the other color than Maricha and Subahu. Clearly the code of the cult is met. For any two people

that know each other, if both were already in the cult then they would have had different colored hats. If one of them is Thelma, then the other must be Maricha or Subahu, and hence will have different colored hats.

Case 2: Maricha and Subahu are wearing different colored hats.

Let us assume Maricha is wearing hat of color x and Subahu is wearing hat of color y , where x and y are different. Let S be the set of people Maricha indirectly knows currently (before Thelma joins the group). We know that Subahu is not in this set. Let us now flip the hat colors of people in S only. Note that if two people know each other and one of them is in S , then the other is also in S (since S is the set of people Maricha knows indirectly). So after switching hat colors, the code of the cult will still be met.

Now, Maricha has a hat of color y , and Subahu's hat color remains y . We can now add Thelma with hat color x , and since Thelma knows no one other than Maricha and Subahu, the cult code is met.

Aside: We could have even accomodated Thelma's wish if she wanted her hat color to be red. We would switch the hats of people Maricha indirectly if Maricha was wearing red. And similarly switch the hats of people Subahu indirectly if Subahy was wearing red. This would ensure both Maricha and Subahu are wearing blue hats before Thelma joins. And then we can add Thelma to the cult and assign her a red hat.

QED.