Functions

Magritte

Discrete Structures (CS 173)

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REMARKS ON PREVIOUS LECTURES

Example Inverse

Definition: y is the *inverse of* x *in* $(mod\ n)$ iff

$$x * y \equiv 1\ (mod\ n)$$

number	0	1	2	3	4	5
inverse	--	1	--	--	--	5

-- inverse does not exist

1. *Does* 0 *have an inverse in* $(mod\ n)$ for any n?
2. *Is the inverse of* 1 *always* 1 in $(mod\ n)$ for any n?

GCD and mod

Claim: $a \equiv b \,(mod\ n) \Rightarrow \gcd(a,n) = \gcd(b,n)$

Proof: $a \equiv b \,(mod\ n)$

$\Rightarrow \exists m \in \mathbb{Z}\,(a = b + m * n)$ definition

$\Rightarrow \gcd(a,n) = \gcd(b,n)$ *(why?)*

Hint: $\gcd(a,n) \mid a$ and $\gcd(a,n) \mid n$

$\Rightarrow \gcd(a,n) \mid (a - m * n)$

$\Rightarrow \gcd(a,n) \mid b$

$\Rightarrow \gcd(a,n) \leq \gcd(b,n)$

Now apply the argument the other way round to show

$\gcd(b,n) \leq \gcd(a,n)$

Hint: $\gcd(b,n) \mid b$ and $\gcd(b,n) \mid m * n$ …

Thus $\gcd(b,n) = \gcd(a,n)$

Inverse in modular arithmetic

Definition: y is the *inverse of* x *in* $(mod\ n)$ iff

$$x * y \equiv 1\ (mod\ n)$$

Some crypto algorithms require finding this inverse.

If an inverse of x in $(mod\ n)$ exists then

$$\gcd(x, n) \equiv 1 \qquad \textit{(why?)}$$

Then by Extended Euclid's algorithm, there exist p and s such that $p\,x + s\,n = 1$

So: $p\,x = 1 + (-s)n$,

or $p\ (mod\ n)$ is the inverse of $x\ (mod\ n)$

Inverse of x in mod and gcd

Claim: If an inverse of x in $(mod\ n)$ exists then $\gcd(x, n) = 1$.
Observe: $a \equiv b\ (mod\ n) \Rightarrow \gcd(a, n) = \gcd(b, n)$

Suppose $x * y \equiv 1\ (mod\ n)$
Then $\gcd(x * y, n) = \gcd(1, n)$

Now $\gcd(1, n) = 1$, so $\gcd(x * y, n) = 1$
We can conclude: $\gcd(x, n) = 1$ *(why?)*

If an inverse of x in $(mod\ n)$ exists **then** $\gcd(x, n) = 1$
is equivalent to
An inverse of x in $(mod\ n)$ exists **only if** $\gcd(x, n) = 1$

Extended Euclidean Algorithm Example

gcd (81,57)

81 = 1 (57) + 24

57 = 2 (24) + 9

24 = 2 (9) + 6

9 = 1(6) + **3**

6 = 2(3) + 0

$\mathbf{3} = 9 - 1(6)$

$\mathbf{3} = 9 - (24 - 2(9))$

$= 3(9) - 1(24)$

$= 3(57 - 2(24)) - 1(24)$

$= 3(57) - 7(24)$

$= 3(57) - 7(81 - 1(57))$

$\mathbf{3} = 10(57) - 7(81)$

The gcd(a,b) can be expressed as a *linear combination* of a and b

Equivalent definitions of Antisymmetry

Antisymmetric: $\forall x, y \in A$ with $x \neq y$, $xRy \rightarrow y\not{R}x$

or equivalently: $\forall x, y \in A,\ xRy \wedge yRx \rightarrow x = y$

Suppose R is antisymmetric:

$$\forall x, y \in A\ (x \neq y \wedge xRy) \rightarrow y\not{R}x$$

Let $xRy \wedge yRx$ be true.

Now yRx implies $y\not{R}x$ is false. Thus $(x \neq y \wedge xRy)$ is false.

But xRy is true, so $x \neq y$ must be false. Thus: $x = y$.

So R antisymmetric implies $\forall x, y \in A,\ xRy \wedge yRx \rightarrow x = y$

Conversely suppose: $\forall x, y \in A,\ xRy \wedge yRx \rightarrow x = y$

Let $(x \neq y \wedge xRy)$.

Now if yRx is true, $xRy \wedge yRx$ is true. So $x = y$.

This contradicts $x \neq y$, so yRx must be false.

Thus: $(x \neq y \wedge xRy) \rightarrow y\not{R}x$ **QED**

Disproof of transitive

Claim: “is square of” is not transitive.

Definition: Relation R on set A is transitive iff $\forall x, y, z \in A, xRy \wedge yRz \rightarrow xRz$

Proof by contradiction.

Suppose “is square of” is transitive.

We know: “81 is the square of 9” and “9 is the square of 3”,

By transitivity we can conclude: “81 is the square of 3”.

But 81 is not the square of 3, so the assumption that “is square of” is transitive is false. **QED**

overhead

Proof of antisymmetric

Claim: “is square of” is antisymmetric.

Definition: Relation R on set A is antisymmetric if $\forall x, y \in A,\ xRy \wedge yRx \rightarrow x = y$

Proof: Let R be “is square of”.

Suppose $xRy \wedge yRx$ holds.

Then $x = y^2$ and $y = x^2$

So $x = (x^2)^2$ or $x = x^4$

i.e., $1 = x^3$ or $x = 1$.

Now $y = x^2$ so $y = 12 = 1$.

Thus $x = y$

We have proved: if $xRy \wedge yRx$ holds then $x = y$.

i.e. R is antisymmetric. **QED**

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Proof of equivalence

Claim: “congruent mod k” is an equivalence relation

Definitions: An equivalence relation is reflexive, symmetric, and transitive.

$$a \equiv b \,(mod\, n) \text{ if } \exists m \in \mathbb{Z}\,(a = b + m\, n)$$

Proof:

(1) Reflexive.

(1) Symmetric

(2) Transitive

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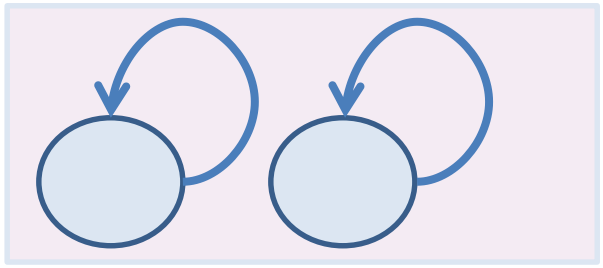
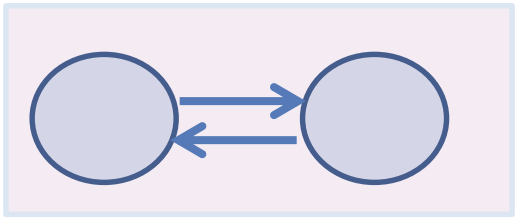
Relations

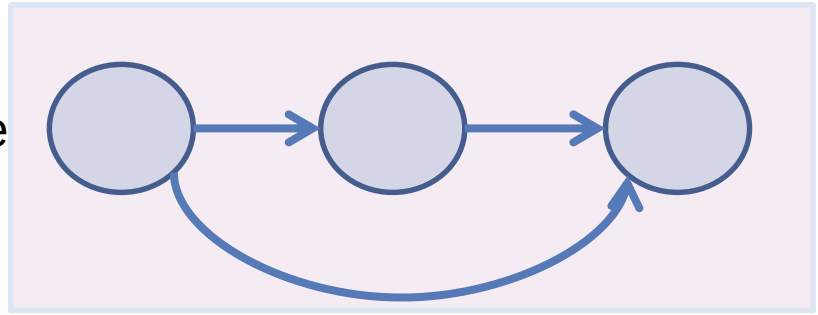
Reflexive

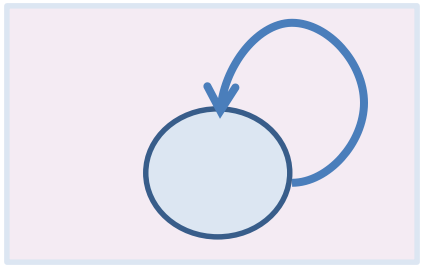


Symmetric



Transitive

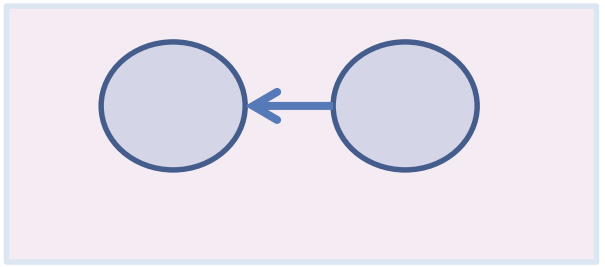




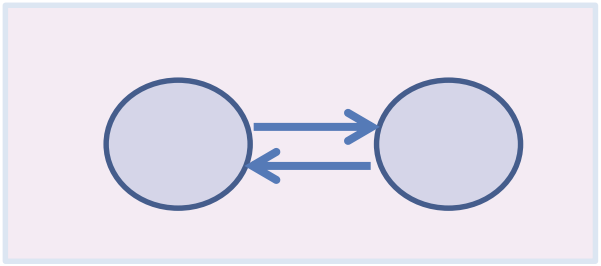
Vacuous truth

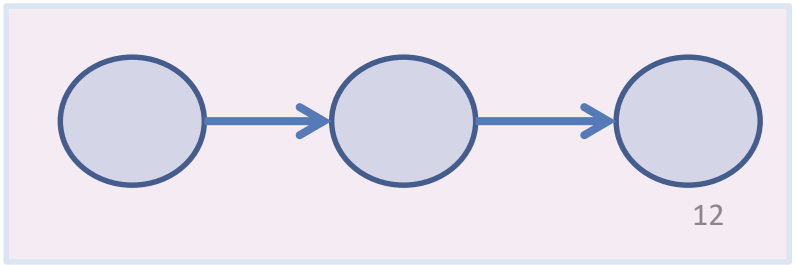
Reflexive, Symmetric and transitive

Irreflexive, Antisymmetric



Not Transitive





Homogeneous and (Heterogeneous) Relations

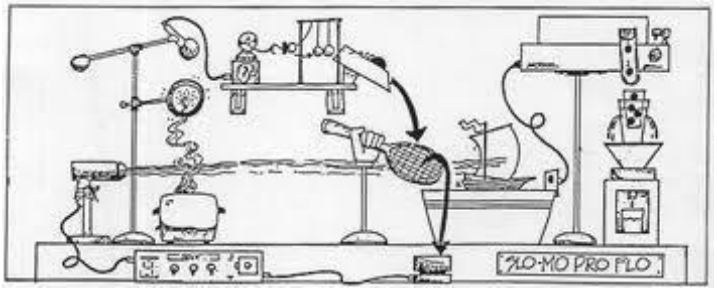
- We defined relations as *homogeneous* relations (i.e., as a subset of *A x A*)
- In general relations can be over any cross-products.
- For any two sets *A* and *B,* a *relation is a subset of A x B*

Some Things to Remember

- How to illustrate a relation graphically
- Be able to identify basic properties of relations: reflexivity, symmetry, transitivity
- Types of relations: partial order, strict partial order, linear order
- When proving (or disproving) a property of a relation, write down definition of relation and property

Functions

- What is a function, and what is not?
- Identify which functions are "*onto*" (or *surjective*)
- Nested quantifiers
  - Mixing "for all" and "there exists"
- Composing functions



What is a function?

Function: is a relation over $A \times B$ which maps each element of A to an element of B, so that an element maps to exactly one element and every element of A is mapped to something, i.e.:

1. $\forall \mathrm{a} \in A \; \exists b \in B, (\mathrm{a}, \mathrm{b}) \in f$
2. $\text{if } (a, b) \in f \text{ and } (a, c) \in f \text{ then } b = c.$

For functions: We write $f(a) = b$ or $f: a \mapsto b$

Functions terminology

$f: A \to B$ stands for f is a map $from\ A\ to$ B

A is the set of '*inputs*'
B is the set of '*outputs*'

Examples of functions

Concepts: mapping; bubbles, plots

Functions: age, t-shirt color, x^2

Functions may be *typed.* In this case, they have a *type signature.*

(In many programming languages, functions are typed).

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What is not a function?

Not a valid function if

1. Some input is not mapped to an output
2. Some input is mapped to two outputs

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When are functions equal?

Functions are equal if

1. They are over the same domain and range.
2. The mapping is the same.

$$f: A \to B \text{ and } g: A \to B \text{ are equal iff}$$
$$\forall a \in A, f(a) = b \text{ iff } g(a) = b$$

Image and Onto

The *image* of a function is the set of values produced when a function is applied to all "inputs"

$$\text{image}(f: A \to B) \subseteq B$$

A function is *onto* or *surjective* if the image is equal to the co-domain (every possible "output" is assigned to at least one input)

$$f: A \to B, \forall y \in B \; \exists x \in A \; f(x) = y$$

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Example:

Claim: $f: R\backslash\{0\} \to R\backslash\{0\}$, $f(x) = 1/x$ is onto.

Recall: $R\ \backslash\{0\}$ is $R\ except\ 0$, i.e. the set of real numbers not including 0.

Definition:

$f: A \to B, f(x)$ is onto iff $\forall y \in B\ \exists x \in A\ f(x) = y$

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Proof of onto

Claim: $f: N^2 \to Z$, $f(x, y) = x - y$ is onto.

Definition: $f: A \to B, f(x)$ is onto iff $\forall y \in B \; \exists x \in A \;\; f(x) = y$

Next Lecture..

- Functions and more functions