CS 173: Fall 2014, A lecture
Long-form homework 3

For this homework, you will submit a solution to one of the following problems via moodle. See moodle for which problem you have been assigned. It is due at 11:45pm on Friday, November 7th..

You must do this problem by yourself. You may not work with classmates.

The grading rubric will place a heavy emphasis on style and logical order. When the instructions for a problem specify a particular proof technique, you must use that method even if the proof could have been done in other ways. If you're solving a problem involving objects such as graphs, your solution must relate any variables and equations to the objects (e.g. n is the number of nodes in the graph).

Problem 1

Use induction to prove the following claim:

> Claim: For any positive integer n, if G is a graph with $2n$ nodes and G has no 3-cycles, then G has $\leq n^2$ edges.

Problem 2

Use induction to prove that

$$(3+\sqrt{5})^n + (3-\sqrt{5})^n \text{ is an even integer for all natural numbers n}$$

Hint: Notice that this expression has the form $x^n + y^n$. Ignore what x and y happen to be in this case. Instead, for a general x and y, figure out how to express $x^{n+1} + y^{n+1}$ in terms of $x^n + y^n$, $x^{n-1} + y^{n-1}$, $x + y$, and xy. Then figure out what $x + y$ and xy are for this specific problem.

Problem 3

A *spanning tree* of a graph G is a connected subgraph that includes all the nodes of G (but perhaps not all the edges) and has no cycles.

Let us define a function g as follows:

$$g(0) = 1$$
$$g(n) = 2^{n-1}(g(n-1))^2, \text{ for } n \geq 1$$

Use induction to prove the following claim:

> Claim: For all natural numbers n, the hypercube Q_n contains at least $g(n)$ spanning trees.

Hint 1: The structure of your inductive proof should follow the recursive structure of the definition of Q_n.

Hint 2: Recall that Q_n is constructed from two copies of Q_{n-1}. Consider the process of building a spanning tree for Q_n by combining spanning trees for the two copies of Q_{n-1}. How many choices do you have for each part of this construction?

Hint 3: Notice that you are proving only a lower bound, not an exact result. So you don't need to figure out how to generate all possible spanning trees. You just need to figure out how to generate enough spanning trees.

Problem 4

Use induction to prove the following claim:

> For any graph G and any two nodes a and b in G, if there is a walk from a to b, then there is a path from a to b.

In your proof, you should represent a walk from a to b as a sequence of nodes e.g. $a = n_1, n_2, \ldots, n_p = b$.

Hint: Induction variables measure the size of something. What's a sensible definition of the size of a path?

Hint: When splitting a big object (e.g. a graph) into a smaller object plus some extra stuff (e.g. one or more extra nodes), you get to pick how to do the split (e.g. which node(s) to remove from a graph). You're allowed to make a choice with some special properties if that helps you write the inductive step.